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On a problem of hypoellipticity

Dedicated to Professor SIGERU MIZOHATA on his sixtieth birthday

By

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§ 1. Introduction.

In the work [11], L. Schwartz has introduced the notion of *hypoellipticity*, and proposed the following question (see p. 146, *Remarques* 2°).

Let $a(x, D)$ be a differential operator, and suppose that it has the following property: there is a positive integer l such that every C^l solution $u(x)$ of

$$a(x, D)u(x)=0$$

belongs to C^∞. Then, can we claim that $a(x, D)$ is hypoelliptic?

We reformulate his question in the following manner.

Let $P(x, D)$ be a differential operator of order $m\geqq 1$ in an open set Ω in $\boldsymbol{R}^n$, with infinitely differentiable coefficients.

Problem I. *Assume that $P(x, D)$ has the following property: given any open subset ω of Ω, there is an integer $l\geqq m$ such that every C^l solution $u(x)$ of*

$$P(x, D)u(x)=0 \tag{1}$$

in ω belongs to $C^\infty(\omega)$. Then, is $P(x, D)$ hypoelliptic in Ω?

Problem II. *Let P_0 be a point of Ω. Assume that $P(x, D)$ has the following property: there exist an integer $l\geqq m$ and a neighborhood $\mathcal{U}$ of P_0 such that every C^l solution $u(x)$ of* (1) *in $\mathcal{U}$ belongs to $C^\infty(\mathcal{U})$. Then, is $P(x, D)$ hypoelliptic at P_0?*

Here $P(x, D)$ is said to be *hypoelliptic at* P_0 if there is a neighborhood $\mathcal{V}$ of P_0 such that, given any distribution u in $\mathcal{V}$, u is a C^∞ function in every neighborhood of P_0 where this is true of $P(x, D)u$ (we recall that $P(x, D)$ is said to be hypoelliptic in Ω if, given any distribution u in Ω, u is a C^∞ function in every open set where this is true of $P(x, D)u$).

We know that both of Problems I and II are positive when the coefficients of $P(x, D)$ are constants. But we observe that these are negative in general in case of variable ones. In fact, let $P(x, D)=x^\alpha P_0(x, D)$, where α is an arbitrary complex constant such that $|\alpha|\geqq m+1$ and $P_0(x, D)$ is an arbitrary elliptic

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