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# Some Numerical invariants of hyperelliptic fibrations

By

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## 0. Introduction

In this paper we shall define certain numerical invariants of hyperelliptic fibrations and study their properties. A proper surjective morphism $\Pi$: $X \to C$ is called a *hyperelliptic fibration of genus* $g$ if $X$ is a smooth surface, $C$ is a smooth curve and the general fiber of $\Pi$ is a hyperelliptic curve of genus $g$.

Hyperelliptic fibrations of genus 2 have been studied by several mathematicians. Ueno [$U$] and Xiao [$X$] proved that the *topological index* $i_{top}(x) = \frac{1}{3}(c_1(X)^2 - 2c_2(X))$ is non-positive if $X$ has a hyperelliptic fibration of genus 2 over a smooth compact curve.

In this paper we will show the following inequalities for every relatively minimal hyperelliptic fibration $\Pi$: $X \to C$ over a smooth compact curve:

$$\frac{-g-1}{2g+1} \cdot \sum_{t \in C} e_t(X) \leq i_{top}(X) \leq \frac{g^2-2g-1}{2g+1} \cdot \sum_{t \in C} e_t(X) \qquad \cdots (0.1.1)$$

*if $g$ is even,*

$$\frac{-g-1}{2g+1} \cdot \sum_{t \in C} e_t(X) \leq i_{top}(X) \leq \frac{g^2-2g}{2g+1} \cdot \sum_{t \in C} e_t(X) \qquad \cdots (0.1.2)$$

*if $g$ is odd.*

where $e_t(X) = (\textit{Euler number of } \Pi^{-1}(t)) - (2-2g)$. (See Theorem 4.0.1 below).

To prove these inequalities, we need a section $D$ of $(\overset{g}{\Lambda}\, \Pi_* \omega_{X/C})^{\otimes 4(2g+1)}$.

For every hyperelliptic fibration $\Pi$: $X \to C$, there exist a $\mathbf{P}^1$-bundle $p$: $Y \to C$ and a double covering $\hat{\Pi}$: $\hat{X} \to Y$ such that $\hat{X}$ is birational to $X$ over $C$. There exists an open set $C^0$ in $C$ which satisfies the following:

i) $p^{-1}(C^0)$ is isomorphic to $\mathbf{P}^1 \times C^0$;

ii) $\Pi^{-1}(C^0)$ can be identified with the closure of $\{(x, t, y) \in \mathbf{C} \times C \times \mathbf{C};\ y^2 = \varphi(x, t)\}$, where $x$ is an inhomogeneous coodinate of $\mathbf{P}^1$ and $\varphi$ is a polynomial of $x$ of degree $2g+1$ or $2g+2$ with coefficients in the rational function field of $C^0$.

The section $D$ of $(\overset{g}{\Lambda}\, \Pi_* \omega_{X/C})^{\otimes 4(2g+1)}$ is defined to be

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