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On elliptic cyclopean forms

By

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Introduction

Shimura [3] defined cyclopean forms and showed that they are closely related to the zeros of L-functions by giving the necessary and sufficient condition on the existance of cyclopean forms. The purpose of this paper is to give generators of the space of cyclopean forms in the elliptic modular case and to prove that the examples he gave in [3] exhaust all cyclopean forms.

Let Γ be a congruence modular group. An automorphic eigenform is a function on the upper half complex plane $\mathbf{H}$ which is an eigen function of the differential operator $L_k = -4y^2 \dfrac{\partial^2}{\partial z \partial \bar{z}} + 2iky \dfrac{\partial}{\partial \bar{z}}$ and satisfies the automorphic condition (1.2a) and the magnitude condition (1.2c) at cusps. We denote by $\mathscr{A}_k(\Gamma, \lambda)$ the space of automorphic eigenforms belonging to the eigenvalue λ. We say f $(\in \mathscr{A}_k(\Gamma, \lambda))$ is a cusp form if it satisfies (1.2d) and denote by $\mathscr{S}_k(\Gamma, \lambda)$ the subspace of $\mathscr{A}_k(\Gamma, \lambda)$ consisting of all cusp forms. We also denote by $\mathscr{N}_k(\Gamma, \lambda)$ the orthogonal complement of $\mathscr{S}_k(\Gamma, \lambda)$ in $\mathscr{A}_k(\Gamma, \lambda)$ with respect to the Petersson inner product defined by (1.3). Any element $f(z)$ $(\in \mathscr{A}_k(\Gamma, \lambda))$ has a Fourier expansion of the form

$$\begin{aligned} f(z) &= cy^{s_0} + dy^{1-k-s_0} \\ &\quad + \sum_{n=1}^{\infty} a_n \omega(4\pi ny/T;\, k+s_0,\, s_0) e^{2\pi i nz/T} \\ &\quad + y^{-k} \sum_{n=1}^{\infty} b_n \omega(4\pi ny/T;\, s_0,\, k+s_0) e^{-2\pi i n\bar{z}/T}, \end{aligned}$$

where $\omega(t; \alpha, \beta)$ is the Whittaker function (see § 1), $y = \mathrm{Im}(z)$ and s_0 is a complex number satisfying $\lambda = s_0(1-k-s_0)$. We call $f(z) \in \mathscr{N}_k(\Gamma, \lambda)$ a *cyclopean form* if

(c1) $$\lambda = s_0(1-k-s_0) \quad \text{with} \quad -\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2},$$

(c2) the term cy^{s_0} of the Fourier expansion of $f|_k\gamma$ vanishes for all $\gamma \in SL_2(\mathbf{Z})$.

Here "$|_k\gamma$" indicates the operation of γ to functions on $\mathbf{H}$ defined by (1.1). We denote by $\mathscr{C}_k(\Gamma, \lambda)$ the subspace of $\mathscr{N}_k(\Gamma, \lambda)$ consisting of all cyclopean forms. To discuss generators of $\mathscr{C}_k(\Gamma, \lambda)$, we have only to consider the case where $\Gamma = \Gamma(N)$, since $\mathscr{N}_k(\Gamma, \lambda) \subset \mathscr{N}_k(\Gamma', \lambda)$ if $\Gamma \supset \Gamma'$. For a Dirichlet character χ defined modulo