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Equations of evolution on the Heisenberg group I

By

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1. Introduction

Theory of nilpotent Lie groups and its irreducible unitary representations become a powerful tools in the study of linear partial differential operators. (Folland, Helffer-Nourrigat, Rockland, Rothschild, Rothschild-Stein, etc.) The major concern of these works is to study the hypoellipticity or local solvability of linear partial differential operator. We believe that the same spirit is effective in investigating the Cauchy problem for the equations of evolution.

For the fundamental solution of operators on the Heisenberg group, there are also many works. B. Gaveau studied the heat equation and A.L. Nachman investigated the wave equation. Contrary to these works, we are concerned with the well-posedness for the Cauchy problem for the operators of higher order on the Heisenberg group. We hope that this becomes a model case for more general differential operators with multiple characteristics.

In this paper, we shall limit ourselves to treating the parabolic case. Let us consider the operators of higher order on the Heisenberg group $\boldsymbol{H}^n$.

$$P=\partial_t^m+\sum_{j=1}^{m} A_j\partial_t^{m-j},$$

where A_j are the homogeneous right invariant differential operators of order pj on $\boldsymbol{H}^n$. ($p\in\boldsymbol{N}$)

Roughly speaking, our main result is formulated as follows. If for any non-trivial irreducible unitary representation π of $\boldsymbol{H}^n$, $\pi(P)$ satisfies "parabolic" conditions, then the Cauchy problem

$$\begin{cases} Pu=f \\ \operatorname{supp}_t u\subset[0,\infty) \end{cases} \tag{1.1}$$

is well-posed: i.e. for any positive number T and any $f\in C_0^\infty((-T, T)\times\boldsymbol{H}^n)$ with support contained in $[0, T)\times\boldsymbol{H}^n$, there is a solution $u(x, t)\in C^\infty((-T, T)\times\boldsymbol{H}^n)$ of (1.1) and this solution is unique in the Sobolev space subordinated to $\boldsymbol{H}^n$.

2. Statement of results

We recall some notion on the Lie group. (c.f. Rockland [R]) Let G be a simply-connected nilpotent Lie group, with Lie algebra $\mathcal{G}$ and (complexified) universal envelop-

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