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On calculation of $L_K(1, \chi)$ for some Hecke characters

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§ 1. Introduction

Let $L_K(s, \chi)$ be the Hecke L-function for a non-trivial character χ of an ideal class group of an algebraic number field K. The purpose of the present paper is to express $L_K(1, \chi)$ in a form effective for numerical calculation by computers in the case where K and χ satisfy the following two conditions:

(1.1) $\begin{cases} \text{(i) } K \text{ is a quadratic extension field of a totally real algebraic number field and } K \text{ has exactly two real places.} \\ \text{(ii) } \chi \text{ ramifies at all the two real places.} \end{cases}$

The methods of ours are the same as [Ko], [Ka] and [G], i.e., a generalization of the classical Kronecker's limit formula, and Hecke's method described in § 3 and § 5 of [Si1]. However, we calculate, rather than $L_K(1, \chi)$ itself, a suitable coefficient $\kappa_K(C)$ of $\zeta_K(s, C)$ in the Taylor expansion at $s=0$, where $\zeta_K(s, C)$ is the zeta function of an ideal class C of K (see (2.1)). In fact, it suffices to obtain $\kappa_K(C)$ for our purpose, so the statements will be described for $\kappa_K(C)$.

Stark-Shintani conjecture, which is one of our motivation of the present paper, predicts a kind of arithmeticity for $L_K(1, \chi)$ ([St1], [St2], [Sh5] and [T]). However, the conjecture has been unsolved yet except some special cases. In fact, the case where K has exactly one imaginary place and χ ramifies at all real places, which is described in [St2], is unsolved one. If K is of degree 4 over $\boldsymbol{Q}$, this case is contained in the case (1.1) and one can use our results to give numerical datas in the case.

Let us explain the contents in more detail. In § 2, we recall some facts on the order of $\zeta_k(s, C)$ at $s=0$ for a general algebraic number field k and prove a kind of transformation formula for $\zeta_k(s, C)$ (Proposition 1). It plays the same role as the formula (92) in [Si1, p. 140], which has been used in [Ko], [Ka] and [G]. By virture of this formula, we can despense with Gauss sums and the assumption on the primitivity of χ, so the value $\kappa_K(C)$ seems to be the more natural in our computation than $L_K(1, \chi)$. In § 3, we define a harmonic Hilbert modular function $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ in (3.1), which is a generalization of the

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