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An increasing union of q-complete manifolds whose limit is not q-complete

By

VIOREL VÂJÂITU

In this short note, modeling on the beautiful example of Fornæss [2], we produce, for any given integer $q \geq 1$, a complex manifold M which *is an increasing union of q-complete open submanifolds $\{M_\nu\}_{\nu\in\mathbf{N}}$ such that M itself fails to be q-complete.*

For $q=1$, we regain Fornæss' example, however, with a different proof.

To proceed, we recall some definitions [1].

Let M be a complex manifold (always countable at infinity) of dimension n. A function $\varphi\in C^\infty(M, \mathbf{R})$ is said to be q-convex if the Levi form of φ computed in local coordinates has at least $n-q+1$ strictly positive eigenvalues.

M is said to be q-complete (resp. q-convex) if it carries a smooth exhaustion function φ (i. e., such that the sublevel set $\{x\in M \mid \varphi(x)<c\}$ is relatively compact in M for any $c\in\mathbf{R}$) which is q-convex on the whole space M (resp. outside a compact subset of M).

A typical situation in our set-up is :

Example 1. *Let L be a linear subspace of codimension q of the complex projective space $\mathbf{P}^n$. Then $\mathbf{P}^n-L$ is q-complete.*

Proof. Indeed, without any loss of generality, we may take $L=\{w_{p+1}=\cdots=w_n=0\}$, $p:=n-q$, where $[w_0 : \dots : w_n]$ are the homogeneous coordinates on $\mathbf{P}^n$. We check that $\varphi : \mathbf{P}^n-L\to\mathbf{R}$ given by

$$\varphi(w):=\log\frac{|w_0|^2+\cdots+|w_n|^2}{|w_{p+1}|^2+\cdots+|w_n|^2}, \qquad w\in\mathbf{P}^n-L$$

is q-convex and exhaustive.

Since the exhaustion property is obvious, it remains to show the q-convexity. To verify this pass to non-homogeneous coordinates and check that the function $\psi : \mathbf{C}^n\to\mathbf{R}$ by

$$\psi(z)=\log\frac{1+|z_1|^2+\cdots+|z_n|^2}{1+|z_{p+1}|^2+\cdots+|z_{n-1}|^2}, \qquad z\in\mathbf{C}^n$$

is q-convex. But this is quite simple! To see this, we let $F\subset\mathbf{C}^n$ be the complex

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