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Homology and cohomology of Lie superalgebra $\mathfrak{sl}(2, 1)$ with coefficients in the spaces of finite-dimensional irreducible reprsentations

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Introduction

In this paper we give a method of calculation for homology groups and cohomology groups with coefficients in a space of a representation of a Lie superalgebra, and carry out the calculation to determine them for finite-dimensional irreducible $\mathfrak{sl}(2, 1)$-modules.

A motivation of this work is to understand to what extent the idea of the cohomological induction is useful for representations of Lie superalgebras. In the case of reductive Lie algebras, the theory of cohomological induction was introduced by Vogan and Zuckerman and was developed by Vogan, Knapp and others (see e.g. [8]). Studies of the structures of cohomologically induced modules, vanishing theorems and Blattner's multiplicity formula are very important, and there, the Poincaré duality plays a decisive role.

In the case of Lie superalgebras, Chemla [2] proved a Poincaré duality under a restrictive condition that the representation in question has a finite projective dimension. However, we do not know if this restrictive condition is really neccessary and when it holds. In author's previous work [13] (see also [15]), we see that Poincaré duality does not hold, except the trivial case, for finite-dimensional representations of $\mathfrak{gl}(1, 1)$. This situation is also true for the present case of $\mathfrak{sl}(2, 1)$.

In [6], Kac constructed all finite-dimensional irreducible representations of Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(m, n)$ as the quotient module $V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda)$ of a standard induced module $\bar{V}(\Lambda)$ by its maximal proper submodule $I(\Lambda)$. Furutsu studied these modules in detail for $\mathfrak{sl}(2, 1)$ and $\mathfrak{sl}(3, 1)$ in [5]. In the case of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(2, 1)$, $\bar{V}(\Lambda)$ is constructed starting from an irreducible highest weight $\mathfrak{g}_{\bar{0}}$-module with highest weight Λ. According to $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathbf{C} \cdot C$, the highest weight Λ is given as $\Lambda = (\lambda, c)$, where $\lambda \in \mathbf{Z}_{\geq 0}$ is a highest weight for $\mathfrak{sl}(2, \mathbf{C})$ and c is a scalar for C which is in the center of $\mathfrak{g}_{\bar{0}}$. We can construct all finite-dimensional irreducible modules of $\mathfrak{sl}(2, 1)$ quite explicitly. Further, we find that any such irreducible module is equivalent to one of $\bar{V}(\Lambda)$ or $I(\Lambda)$, and that as $\mathfrak{g}_{\bar{0}}$-modules $\bar{V}(\Lambda)$ is a direct sum of four

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