J. Math. Kyoto Univ. (JMKYAZ)
39-2 (1999) 331–346

Tauberian theorem of exponential type and its application to multiple convolution

By

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1. Introduction

Let $\{U_n(x)\}$ be a sequence of non-decreasing, right-continuous functions on $\mathbf{R}$ vanishing on $(-\infty, 0]$, and let $\omega_n(s)$ be the Laplace transform of $U_n(x)$. In this paper, we shall study the relationship between the asymptotic behavior of $\log \omega_n(ns)$ and that of $\log U_n(x)$. This problem is motivated by the following question: Let $X_1, X_2, \ldots$ be positive, independent random variables with common distribution. By the law of large numbers, we see that $X_1 + X_2 + \cdots + X_n \to \infty$ as $n \to \infty$, *a.s.*, and it is of interest to know how fast $P[X_1 + X_2 + \cdots + X_n \leq a](a > 0)$ tends to 0 as $n \to \infty$. In other words, we are interested in the asymptotic behavior of the multiple convolution

$$\int \cdots \int_{0 < x_1 + x_2 + \cdots + x_n \leq a} d\sigma(x_1) d\sigma(x_2) \cdots d\sigma(x_n), (a > 0)$$

as $n \to \infty$, where $\sigma(x)$ is a right-continuous non-decreasing function vanishing on $(-\infty, 0]$ and here we no longer need to assume that σ is a distribution function. This may be considered as a problem of large deviation. In a study of the local time of Gaussian processes, Kasahara, *et al.* obtained the following result (Lemma 3 of [11]). If σ varies regularly at 0 (i.e., $\lim_{\lambda \to 0} \sigma(\lambda x)/\sigma(\lambda) = x^\alpha, x > 0$, for some α; see [1]), then

$$\left(\int \cdots \int_{0 < x_1 + \cdots + x_n \leq 1} d\sigma(x_1) \cdots d\sigma(x_n) \right)^{1/n} \asymp \sigma\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{1.1}$$

where $f \asymp g$ means $0 < \liminf f(x)/g(x) \leq \limsup f(x)/g(x) < \infty$. Our question is to determine the exact constant in (1.1). To this end, we first consider the case of $\sigma(x) = x_+^\alpha (\alpha > 0)$, where $x_+ = x \vee 0$. An elementary calculus provides us with

$$\int \cdots \int_{0 < x_1 + \cdots + x_n \leq 1} d\sigma(x_1) d\sigma(x_2) \cdots d\sigma(x_n) = \frac{\Gamma(\alpha + 1)^n}{\Gamma(\alpha n + 1)}, \tag{1.2}$$

Communicated by Prof. K. Ueno, September 2, 1998