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Inductive limits of topologies, their direct products, and problems related to algebraic structures

By

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Introduction

This paper is a continuation of our previous work [9]. In a half of it, we studied the inductive limit $G = \lim_{\to} G_n$ of topological groups $G_n, n \geq 1$, and proved that the inductive limit topology $\tau_{ind}^G = \lim_{\to} \tau_{G_n}$ of topologies τ_{G_n} on G_n does not in general give a group topology on G, contrary to the affirmative statement in [4, Article 210], and then studying τ_{ind}^G in detail, we constructed, under a mild condition (PTA), a group topology $\tau_{BS}^G = \text{BS-}\lim_{\to} \tau_{G_n}$ on G, called Bamboo-Shoot topology, which is the strongest among those weaker than or equal to τ_{ind}^G. This work provokes us two directions of study.

The one is to study the reason why this kind of pathological phenomena occur rather in general. The other is to construct a good version of inductive limits (like τ_{BS}^G in the category of topological groups) in various categories, such as topological algebras, topological semigroups etc.

Take two inductive systems of topological spaces $\{X_\alpha\}_{\alpha \in A}$ and $\{Y_\alpha\}_{\alpha \in A}$ and put $X = \lim_{\to} X_\alpha, Y = \lim_{\to} Y_\alpha$. For the direct product $\{X_\alpha \times Y_\alpha\}_{\alpha \in A}$ of these systems, its inductive limit can be identified with $X \times Y$, and on it we have two kinds of topologies, the one is $\tau_{ind}^X \times \tau_{ind}^Y$ with $\tau_{ind}^X := \lim_{\to} \tau_{X_\alpha}$ and the other is $\tau_{ind}^{X \times Y} := \lim_{\to}(\tau_{X_\alpha} \times \tau_{Y_\alpha})$. Then, we have in general $\tau_{ind}^X \times \tau_{ind}^Y \preceq \tau_{ind}^{X \times Y}$. We found that a principal reason for pathological phenomena similar to the above one is the mismatch of these two topologies on $X \times Y$. Therefore we propose, in Section 1, Problems A, B and C related to these phenomena, and study them in Sections 4 through 6.

Assume $\{X_\alpha\}_{\alpha \in A}$ be an inductive system in the category of locally convex topological vector spaces (= LCTVSs). Then, the natural inductive limit topology in this category (cf. Definition 2.1) has been given long ago (denoted here as $\tau_{lcv}^X = \text{lcv-}\lim_{\to} X_\alpha$) and is now used everywhere. As an example, take a space of test functions $\mathcal{D}(M)$ of C^∞-functions with compact supports on a

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