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On the holomorphic rank-2 vector bundles with trivial discriminant over non-Kähler elliptic bundles

By

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1. Introduction

Let X be a smooth connected compact complex surface. A classical problem, in its simplified form, is to determine the pairs $(c_1, c_2) \in NS(X) \times \mathbf{Z}$, for which there exists a holomorphic rank-2 vector bundle E on X, having Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$.

For projective surfaces, the problem was solved by Schwarzenberger (cf. [13]): in this case, for any pair (c_1, c_2), there exists a smooth double covering $Y \xrightarrow{\varphi} X$, and a line bundle L over Y, such that $c_1(\varphi_* L) = c_1$ and $c_2(\varphi_* L) = c_2$.

In contrast to this situation, for non-projective surfaces, there is a natural *necessary* condition for the existence problem (cf. [2, Theorem 3.1], [6, Proposition 1.1], [8]):

$$\Delta(c_1, c_2) := \frac{1}{2}\left(c_2 - \frac{c_1^2}{4}\right) \geq 0.$$

In order to be able to find sufficient conditions for this existence problem, one tries to use the classical known methods for constructing vector bundles having given Chern classes c_1 and c_2. Serre's method, by far the most productive, brings out vector bundles whose Chern classes satisfy the inequality: $\Delta(c_1, c_2) \geq m(c_1)$ (cf. [2], see also [6], [11]), where

$$m(c_1) := -\frac{1}{2} \sup_{\mu \in NS(X)} \left(\frac{c_1}{2} - \mu\right)^2.$$

By using coverings of surfaces, we produced holomorphic vector bundles on primary Kodaira surfaces whose discriminants satisfy $0 \leq \Delta(c_1, c_2) < m(c_1)$, as direct images of line bundles (cf. [1], see also [15]). Unfortunately, this method is not very extensive and it is the aim of this Note to investigate its limits. We consider the case when X is a non-Kähler elliptic bundle over a curve B of genus at least 2. If $C \xrightarrow{\pi} B$ is a double covering, and $Y := C \times_B X \xrightarrow{\varphi} X$ denotes

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