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KO-theory of flag manifolds

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1. Introduction

The purpose of this paper is to determine the KO^*-groups of flag manifolds which are the homogeneous spaces $G(n)/T$ for $G = U, Sp, SO$ and T is the maximal torus of $G(n)$. We compute it by making use of the Atiyah-Hirzebruch spectral sequence and obtain the following.

Theorem. *The KO^i-groups of $G(n)/T$ for $G = U, Sp, SO$ are as in Table 1, where $s = n!/2, 2^{n-1}n!$ for $G = U, Sp$ and $s = 2^{m-2}m!, 2^{m-1}m!$ for $G = SO$ and $n = 2m, 2m+1$ respectively.*

2. The Atiyah-Hirzebruch spectral sequence

First we recall that the coefficient ring of KO-theory is that

$$KO^* = \mathbf{Z}[\alpha, x, \beta, \beta^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2 - 4\beta),$$

where $|\alpha| = -1$, $|x| = -4$ and $|\beta| = -8$.

Let X be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is the spectral sequence with $E_2^{p,q} \cong H^p(X; KO^q)$ converging to $KO^*(X)$. It is well known that the differential d_2 of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is given by the following (see [2]).

$$d_2^{*,q} = \begin{cases} Sq^2\,\pi_2, & q \equiv 0\ (8), \\ Sq^2, & q \equiv -1\ (8), \\ 0, & \text{otherwise}, \end{cases}$$

where π_2 is the modulo 2 reduction.

It is well known that G/T is a CW-complex with only even cells, where G is a compact connected Lie group and T is the maximal torus of G ([1]). The next proposition, given in [4] and [5], is concerned with the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ for the special X which can be G/T.

Proposition 2.1. *Let X be a CW-complex whose cohomology is torsion free and concentrated in even dimension, and $E_r(X)$ be the r-th term of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$. Then we have the following.*

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