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Logarithmic sheaves attached to arrangements of hyperplanes

By

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1. Introduction

Any divisor D on a nonsingular variety X defines a sheaf of logarithmic differential forms $\Omega_X^1(\log D)$. Its equivalent definitions and many useful properties are discussed in a fundamental paper of K. Saito [Sa]. This sheaf is locally free when D is a strictly normal crossing divisor, and in this situation it is a part of the logarithmic De Rham complex used by P. Deligne to define the mixed Hodge structure on the cohomology of the complement $X \setminus D$. In the theory of hyperplane arrangements this sheaf arises when D is a central arrangement of hyperplanes in $\mathbb{C}^{n+1}$. In exceptional situations this sheaf could be free (a free arrangement), for example, when $n = 2$ or the arrangement is a complex reflection arrangement. Many geometric properties of the vector bundle $\Omega_X^1(\log D)$ were studied in the case when D is a generic arrangement of hyperplanes in $\mathbb{P}^n$ [DK1]. Among these properties is a Torelli type theorem which asserts that two arrangements with isomorphic vector bundles of logarithmic 1-forms coincide unless they osculate a normal rational curve. In this paper we introduce and study a certain subsheaf $\tilde{\Omega}_X^1(\log D)$ of $\Omega_X^1(\log D)$. This sheaf contains as a subsheaf (and coincides with it in the case when the divisor D is the union of normal irreducible divisors) the sheaf of logarithmic differentials considered earlier in [CHKS]. Its double dual is isomorphic to $\Omega_X^1(\log D)$. The sheaf $\tilde{\Omega}_X^1(\log D)$ is locally free only if the divisor D is locally formally isomorphic to a strictly normal crossing divisor. This disadvantage is compensated by some good properties of this sheaf which $\Omega_X^1(\log D)$ does not posses in general. For example, one has always a residue exact sequence

$$0 \to \Omega_X^1 \to \tilde{\Omega}_X^1(\log D) \to \nu_*\mathcal{O}_{D'} \to 0,$$

where $\nu : D' \to D$ is a resolution of singularities of D. Also, in the case when D is an arrangement of m hyperplanes in $\mathbb{P}^n$, the sheaf $\tilde{\Omega}_{\mathbb{P}^n}^1(\log D)$ admits a simple projective resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}_{\mathbb{P}^n}^{m-1} \to \tilde{\Omega}_{\mathbb{P}^n}^1(\log D) \to 0.$$

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