ON A CONJECTURE OF MILNOR

BY

K. VARADARAJAN

Introduction

Let X denote a connected space dominated by a finite CW-complex and let $\pi = \pi_1(X)$. In his paper *Finiteness conditions for CW-complexes* [6], C. T. C. Wall associates to X an element $\sigma(X) \in \tilde{K}_0\ (\mathbf{Z}(\pi))$, called the obstruction to finiteness of X, whose vanishing is necessary and sufficient for X to be of the homotopy type of a finite complex. Also Wall shows that given any finitely presentable group π and any $\sigma \in \tilde{K}_0\ (\mathbf{Z}(\pi))$ there exists a CW- complex X dominated by a finite complex and satisfying $\pi_1(X) \approx \pi; \sigma(X) = \sigma$.

Any compact topological manifold is a retract of a finite complex. As such to each connected compact manifold M is associated an element $\sigma(M) \in \tilde{K}_0(\mathbf{Z}(\pi))$ (where $\pi = \pi_1 M$) which is the obstruction to finiteness of M. It is a conjecture of Milnor that for a connected, closed manifold M^n of dimension n the relation $\sigma(M) = (-1)^n\overline{\sigma(M)}$ holds, where bar denotes the involution in $\tilde{K}_0(\mathbf{Z}(\pi))$ arising from the involution $\sum m_i x_i \rightarrow \sum m_i x_i^{-1}$ of $\mathbf{Z}(\pi)$. Thus if this conjecture is proved it will follow that not every element in $\tilde{K}_0(\mathbf{Z}(\pi))$ can be realised as the Wall obstruction of a closed manifold, where π is an arbitrary finitely presentable group. Siebenmann in his thesis [4] proves this equality with the additional assumption that $M \times \mathbf{R}^k$ carries a differentiable structure for some integer k. The object of this paper is to prove this equality for all closed, orientable manifolds.

The author has learned that the formula $\sigma(M) = (-1)^n\overline{\sigma(M)}$ for a closed orientable manifold M of dimension n has been proved independently by Milnor and Wall and that Milnor's proof will appear in a forthcoming paper of Wall entitled *Poincaré complexes I* to appear in the Annals of Mathematics shortly.[1] This proof is purely algebraic whereas the author's proof is more geometric. Also the author has learned that the result on the Wall obstruction for sphere bundles obtained in this paper has also been obtained by S. Gersten. But none of these has appeared in print at the time of acceptance of this paper.

The idea of the proof can briefly be explained as replacing $M \times \mathbf{R}^k$ in Siebenmann's proof by the total space of an orientable topological $\mathbf{R}^k$ bundle (i.e. a microbundle) over M carrying a differentiable structure. In the course of his proof Siebenmann uses the fact that the Wall obstruction for $M \times S^{k-1}$ is zero when $k - 1$ is odd. For our proof we have to study the Wall obstruction of a sphere bundle over a connected CW-complex X dominated by a finite complex. We have information only in the case of S^k bundles when $k \geqq 2$ (Theorem 3.3). For an S^1 bundle over X we do not have any information.

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[1] This paper of Wall's has already appeared in vol. 86 (1967), pp. 213–245.