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A NOTE ON THE FINITE IMAGES OF PROFINITE GROUPS

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Dedicated to the memory of Reinhold Baer

In [2] and a number of other papers, Reinhold Baer studied the deduction of properties of infinite groups from properties of their finite images. Here we consider one aspect of the inverse problem. Associated naturally with each profinite group G there are two families of finite groups: the family $\mathcal{F}_{\mathrm{o}}(G)$ of quotient groups of G modulo open normal subgroups and the family $\mathcal{F}_{\mathrm{a}}(G)$ of quotients modulo arbitrary subgroups of finite index. The two classes coincide if and only if $\mathcal{F}_{\mathrm{a}}(G)$ is countable (see [7]). While the relationship between G and $\mathcal{F}_{\mathrm{o}}(G)$ is fairly clear, the relationship of $\mathcal{F}_{\mathrm{a}}(G)$ to G and to $\mathcal{F}_{\mathrm{o}}(G)$ is not well understood.

If each group in $\mathcal{F}_{\mathrm{o}}(G)$ satisfies an abstract group law v, then clearly each group in $\mathcal{F}_{\mathrm{a}}(G)$ satisfies v. It was proved by Anderson [1, Theorem 1] that if for some set of primes π each group in $\mathcal{F}_{\mathrm{o}}(G)$ is a π-group then each group in $\mathcal{F}_{\mathrm{a}}(G)$ is a π-group. Moreover if $\mathcal{F}_{\mathrm{o}}(G)$ consists of soluble (resp. nilpotent) groups, then so does $\mathcal{F}_{\mathrm{a}}(G)$; see [1, Proposition 6] for the soluble case. The proofs of these two latter statements are indirect: they rely on Lemma 1 below and the fact that prosolubility (resp. pronilpotency) can be characterized in terms of the existence (resp. existence and normality) of p-complements. Beyond these results, little seems to be known about the influence of $\mathcal{F}_{\mathrm{o}}(G)$ on $\mathcal{F}_{\mathrm{a}}(G)$.

It has been conjectured by Serre that if H is a finitely generated profinite group then each subgroup of finite index is open. If true, this conjecture would imply that each group in $\mathcal{F}_{\mathrm{a}}(G)$ is an epimorphic image of a subgroup of a group in $\mathcal{F}_{\mathrm{o}}(G)$. Indeed, let K be a normal subgroup of finite index in G; let T be a transversal to K in G and let H be the closed subgroup of G generated by T; thus H is a finitely generated profinite group having $H \cap K$ as a normal subgroup of finite index. The conjecture would imply that $H \cap K$ is open in H, so that $H \cap K \geqslant H \cap N$ for some open normal subgroup N of

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