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THE STRONG TEST IDEAL

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Let R be a Noetherian commutative ring containing a field. The *test ideal,* introduced by Hochster and Huneke in [HH1], has emerged as an important object associated to R. The test ideal can be defined as the largest ideal J of R such that $JI^* \subset I$ for all ideals I of R where I^* denotes the tight closure of I. Although it is not obvious that a ring R admits a non-zero test ideal, Hochster and Huneke showed nearly every ring of interest possesses a non-zero test ideal. (The definition of tight closure and basic features of the test ideal are recalled in Section 0.)

In [Hu], Craig Huneke introduced the related concept of a *strong test ideal:* an ideal J of R such that $JI^* = JI$ for all ideals I of the ring, where I^* denotes the tight closure of I. Huneke showed that non-trivial strong test ideals exist for a reasonably large class of rings, and put them to interesting use bounding the degrees of the equations of integral dependence for certain elements in the integral closure of an ideal. He also asked whether the blowup of the maximal strong test ideal might be a variety with only rational singularities, or some other good properties.

The purpose of this paper is to show that in many cases the test ideal is itself a strong test ideal. Since a sum of strong test ideals is a strong test ideal, there exists a unique maximal strong test ideal for R, and it is natural to call it *the* strong test ideal. From the definitions, we see that every strong test ideal is contained in the test ideal, but there is no *a priori* reason to expect them to coincide. Our paper shows that in many cases, the strong test ideal and the test ideal coincide. This provides numerous non-trivial examples in which the strong test ideal, proven to exist but not constructed explicitly by Huneke, can be explicitly described. This allows us to answer questions posed in [Hu].

An outline of the main results of the paper follows. Section 0 reviews basic definitions and properties of tight closure and test ideals.

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