ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 1987

EISENSTEIN SERIES AND CARTAN GROUPS

BY

ISAAC EFRAT[1]

Introduction

The principal congruence subgroup $\Gamma(N)$ acts discontinuously on the upper half plane $\mathscr{H}$, to give a non-compact fundamental domain of finite volume. Given such a group, one can associate to each cusp κ_i an Eisenstein series $E_i(z, s)$, where $z \in \mathscr{H}$ and $s \in \mathbf{C}$. This Eisenstein series admits a Fourier expansion at each cusp κ_j. The zero Fourier coefficient involves a meromorphic function $\phi_{ij}(s)$, so that one obtains a matrix $\Phi(s) = (\phi_{ij}(s))_{i,j}$ (see §1 for precise definitions).

The determinant $\phi(s) = \det \Phi(s)$ plays a key role in the theory, mostly due to its appearance in the Selberg trace formula for the group in question. Of particular importance are the poles of $\phi(s)$, whose analysis is connected with the study of cusp forms for the group (see [11], [1]).

The problem of computing $\phi(s)$ for $\Gamma(N)$ was first addressed by Hejhal (see [4]), who treated the case of square free and odd N by some rather involved methods. Huxley [5] has recently solved the problem using other ingenious arguments, and gave an expression for $\phi(s)$ for any N. As for other groups, we mention the work in [2] where we compute these determinants for Hilbert modular groups, and in [1], where they are partially analyzed for congruence subgroups of Hilbert modular groups. Other relevant references are [3], [8], [9].

Our aim in this paper is to introduce the Cartan group $C(N)$ into the study of the Eisenstein series for $\Gamma(N)$, and to use it in order to give a short and simple proof of the precise formula for $\phi(s)$, for any N. Our main theorem (§3) shows that $\phi(s)$ is naturally expressed in terms of the L-functions on $C(N)$. These L-functions also come up in the work of Kubert and Lang on modular units [7].

1. The Eisenstein series

Let

$$\Gamma = \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I (\operatorname{mod} N) \right\}$$

Received October 11, 1985
[1]Partially supported by a grant from the National Science Foundation.