FOURIER-STIELTJES TRANSFORMS WITH SMALL SUPPORTS

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1. Let G be a locally compact abelian group and S a closed subset of the character group $G^\wedge$. If S is sufficiently "small", it is natural to expect that any finite complex measure μ on G with Fourier-Stieltjes transform $\hat{\mu}$ vanishing off S will be absolutely continuous. As the simplest case, one knows that

(1.1) if S has finite Haar measure, every μ with $\hat{\mu} = 0$ off S is absolutely continuous,

since $\hat{\mu}$ is then integrable [4]. Deeper examples are provided by[2] the F. and M. Riesz theorem (where $G^\wedge$ is the integer group Z and S the non-negative integers) and Bochner's generalization of that result (where $G^\wedge = Z^n$ and S is the positive orthant) [4]. In both these results S has the property that for all $\hat{g}$ in $G^\wedge$

(1.2) $S \cap (\hat{g} - S)$ has finite measure;

the purpose of the present note is to point out that (1.2) alone insures something suggesting absolute continuity, specifically that $\mu * \mu$ is then absolutely continuous for every measure μ with $\hat{\mu}$ vanishing off S.

(In case G is metric, even[3] $|\mu| * |\mu|$ is absolutely continuous. Since there are examples [5] of (non-negative) singular measures μ on the circle group with $\mu * \mu$ absolutely continuous, we are of course still quite far from concluding that (1.2) implies absolute continuity.)

Our proof is mainly measure-theoretic and depends basically on disintegration of measures [1], [2]; just about the only fact from harmonic analysis that is needed is (1.1) itself. Indeed the result comes from the observations that (1.2) says that certain sections of $S \times S$ (by cosets of the antidiagonal of $G^\wedge \times G^\wedge$) have finite measure, and that on each of these sections $(\mu \times \mu)^\wedge$ is the transform of a measure on G which is closely related to $|\mu| * |\mu^*|$—a fact which appears from a disintegration of $\mu \times \mu$.

Since all proofs of the F. and M. Riesz theorem and Bochner's theorem depend (in one way or another) on the fact that there S is a proper subsemigroup of $G^\wedge$, one might hope to obtain the full analogue of these results using such an hypothesis; as will be seen, our proof seems unsuited to producing such a result. However, the approach can be combined with the F.

Received January 25, 1964.

[1] Work supported in part by the National Science Foundation.

[2] In our references to these results below we always have in mind only that half which yields the absolute continuity of μ.

[3] $|\mu|$ denotes the usual absolute value (total variation) measure associated with μ.