ZASSENHAUS' LEMMA ON SECTORIAL NORM-DISTANCES[1]

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1. Introduction

A norm-distance on the Euclidean space, E_n, is a function, say F, from $E_n \times E_n$ to the reals having the properties that for any points P and Q in E_n:

(i) $F(P, Q) \geq 0$.
(ii) $F(P, Q) = F(Q, P)$.
(iii) $F(P + \bar{a}, Q + \bar{a}) = F(P, Q)$ where $\bar{a}$ is any vector in R_n and $P + \bar{a}$, $Q + \bar{a}$ denote respectively the points to which P and Q are translated by $\bar{a}$.
(iv) $F(P, X) + F(X, Q) = F(P, Q)$ where X is any point of the segment PQ.

The translation invariance expressed in (iii) implies that $F(P + \bar{a}, P)$ is independent of P so that $F(P + \bar{a}, P) = f(\bar{a})$ defines a non-negative real-valued function, f, on R_n. $f(\bar{a})$ is called the norm-length of the vector $\bar{a}$. (See for example Cassels [1, Chapter IV].)

In view of (iv), f has the property that

$$f(t\bar{a}) = |t| f(\bar{a})$$

for any real t.

The gauge body of F at P, P a point of E_n, is the set,

$$B(P, F) = \{X \mid X \text{ in } E_n,\ F(P, X) \leq 1\}.$$

It is a star set having P as center of symmetry. If P is an interior point then $B(P, F)$ is called a star body.

In E_2 a packing with respect to F, in the sense of Minkowski-Hlawaka, consists of a finite set of points, E, which is admisible with respect to F (i.e. $F(P, Q) \geq 1$ for any two points P and Q of E) and a Jordan polygon, Π, the vertices of which belong to E and which contains the remaining points of E, if any, in its interior. Such a pair, (Π, E), will also be called an F-distribution.

The term "sectorial norm-distance" has been introduced by Zassenhaus [2] to describe a norm-distance, F, which has the following special property: The complement of $B(0, F)$ consists of a finite and, because of (ii), an even number of disjoint open convex sets $K_1, \cdots, K_{2r}$; each K_i is contained in a sector (i.e. a cone with vertex 0) S_i of E_n ($i = 1, \cdots, 2r$); $\text{int } S_i \cap \text{int } S_j = \emptyset$ if $i \neq j$; $\bigcup S_i = E_n$.

A vector $\bar{a}$ is said to belong to the sector S_i if $0 + \bar{a}$ is in S_i.

A sectorial norm-distance is non-degenerate if and only if $r > 1$. That $r > 1$ will always be assumed in what follows.

In E_2 a sectorial norm-distance gives rise to a classification of triangles into

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