COMMUTATOR CLOSED GROUPS

BY

Hermann Heineken[1]

Introduction

The commutator operation is a useful tool of group theory. However, only very little is known about mappings that carry every element g of G into the commutator $x \circ g$ for fixed x. If the group has the property that a certain power of each of such mappings maps every element onto 1, then G is an engel group, the aspects of which have been investigated by Zorn [1], Levi [1], Baer [1], Gruenberg [1], [2] and the author [1]. We will investigate here another class of groups, namely those groups, where the set of the mappings indicated above is a semigroup with respect to the usual product operation. Actually, corresponding to the different ways of commutator bracketing, we have two definitions:

G is right commutator closed, if for any two elements a, b of the group G there is an element $c \in G$ such that $a \circ (b \circ g) = c \circ g$ holds for all $g \in G$.

G is left commutator closed, if for any two elements a, b of the group G there is an element $c \in G$ such that $(g \circ b) \circ a = g \circ c$ holds for all $g \in G$.

These two classes of groups contain for instance the two-engel groups considered by Levi [1], for in these groups we have the identities

$$x \circ (y \circ z) = (y \circ x) \circ z; \qquad (z \circ y) \circ x = z \circ (x \circ y)$$

which give us directly the element c needed. Unlike the two-engel groups, however, the finite right commutator closed groups may be nilpotent of any class wanted, as Example 1 in Section 5 shows. It will be shown, that right and left commutator closed groups are metabelian (Theorem 1.4). Right commutator closed groups are nilpotent whenever $G/C(G')$ is finitely generated (Theorem 2.1). This may be considered as best possible because there are infinitely generated right commutator closed groups that are neither residually nilpotent nor locally nilpotent (and therefore not Z-A-groups either); see the examples in Section 5. Right commutator closed groups are left commutator closed (Lemma 1.3); see Section 3 for left commutator closed groups that are not right commutator closed.

Not every subgroup of a right (left) commutator closed group is itself right (left) commutator closed; a counterexample is Example 2 in Section 5. So we define the corresponding local properties as follows:

The group G is locally right (left) commutator closed, if any finitely generated

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