THE TYPE SET OF A TORSION-FREE GROUP OF FINITE RANK[1]

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In this paper, we shall show that the type set of a torsion-free group of finite rank has certain lattices of types and of pure subgroups associated with it. Conversely, if certain lattice requirements are met by a finite set of types T and by associated subspaces, then a torsion-free group A can be constructed having type set T. The construction of A suggests defining a class of groups having a similar construction. For this class of groups, we shall next establish a set of quasi-isomorphism invariants, together with several other properties. Finally, we shall examine the structure both of the groups and of the class.

1. Necessary conditions on the type set

DEFINITION 1.1 Throughout this paper, by "group" we shall mean "torsion-free abelian group of finite rank" unless some further qualification is given. Let $\sim$ denote the usual equivalence relation on the set of heights; and let $[h]$ denote the equivalence class, or type, to which the height h belongs. Let $\leq$, $\cap$, and $\cup$ have their usual meaning for both heights and types. The set of all types then forms a distributive lattice in which the meet and join of the types t and t' are given by $t \cap t'$ and $t \cup t'$ respectively, [4, pp. 146–147].

DEFINITION 1.2 Let A be a group of rank n. Use A^* to denote the minimal divisible group containing A. Without loss of generality, it can be assumed that $A \subseteq R^n$ and $A^* = R^n$, where R^n is an n-dimensional rational vector space. Let $0 \neq x \in A$; $t^A(x)$, or simply $t(x)$, denotes the type of x in A. Let $t^A(0) = t_\infty$, a type defined to be greater than all other types. $T(A) = \{t^A(x) \mid x \in A\}$ is called the (augmented) type set of A. Let $C(A) = T(A) \cup \{$all finite intersections of members of $T(A)\}$. $C(A)$ is countable since A is countable.

DEFINTION 1.3 Let t be a type; define $A_t = \{x \in A \mid t(x) \geq t\}$. A_t is a pure subgroup of A, [4, p. 147]. Let

$$P(A) = \{A_t \mid t \in C(A)\} \quad \text{and} \quad P^*(A) = \{A_t^* \mid t \in C(A)\}.$$

We shall use A_k to denote A_{t_k} if no confusion arises.

LEMMA 1.4 *Let A be a group; let $t_1, t_2 \in C(A)$ such that $t_\infty > t_2 > t_1$. Then* $\mathrm{Rank}(A_1) > \mathrm{Rank}(A_2)$.

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