ON THE UNIQUE FACTORIZATION THEOREM IN THE RING OF NUMBER THEORETIC FUNCTIONS

BY

CHIN-PI LU

1. Introduction

The set Ω of all functions $\psi(n)$ on $Z = \{1, 2, 3, \cdots\}$ into a commutative ring R with identity forms a commutative ring with identity under ordinary addition and the multiplication $*$; $(\psi * \chi)(n) = \sum_{d|n} \psi(d) \cdot \chi(n/d)$. It was proved by Cashwell and Everett [2] that when R is the field of complex numbers Ω is a unique factorization domain. In this paper we extend and prove the unique factorization theorem in Ω for a wider class of commutative rings R. The method is indirect and it uses the isomorphism between Ω and the ring of formal power series R_ω in a countably infinite number of indeterminates over R. The theorem is proved for R_ω by introducing a topology.

2. The ring of number theoretic functions

The class Ω of all number theoretic functions ψ, i.e., all functions $\psi(n)$ on the set Z of natural numbers n into a commutative ring with identity forms a commutative ring with identity under the addition $+$,

$$(\psi + \chi)(n) = \psi(n) + \chi(n),$$

and the multiplication $*$ which is called *convolution*,

$$(\psi * \chi)(n) = \sum_{d|n} \psi(d) \cdot \chi(n/d).$$

The zero 0 and the additive inverse $-\psi$ of ψ are of course the functions defined by $0(n) = 0$ and $(-\psi)(n) = -\psi(n)$ for every n. The function E with $E(1) =$ the identity of R, $E(n) = 0$ for all $n \neq 1$, is the identity: $E * \psi = \psi * E = \psi$ for all ψ in Ω. We say that Ω is the ring of number theoretic functions over R if each function of Ω takes values from R. A function $N(\psi)$ on Ω to Z is defined by taking $N(\psi)$ to be the smallest number n for which $\psi(n) \neq 0$ if $\psi \neq 0$ and $N(\psi) = \infty$ if and only if $\psi = 0$. Clearly $N(\psi) \geqq 1$ for all ψ. If R has no zero divisors, then $N(\psi * \chi) = N(\psi) \cdot N(\chi)$ for all ψ, χ of Ω. Indeed, we find that, if $\psi \neq 0$, $\chi \neq 0$ with $N(\psi) = i$ and $N(\chi) = j$, then

$$(\psi * \chi)(i \cdot j) = \sum_{m \cdot n = i \cdot j} \psi(m) \cdot \chi(n) = \psi(i) \cdot \chi(j) \neq 0$$

since $\psi(m) = 0$, $\chi(n) = 0$ for all $m < i$ and $n < j$.

PROPOSITION 1. *The ring Ω of number theoretic functions over a domain of integrity (i.e., a commutative domain with identity) has no zero divisors.*

Received September 3, 1963; received in revised form October 10, 1963.