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IDEALWISE ALGEBRAIC INDEPENDENCE FOR ELEMENTS OF THE COMPLETION OF A LOCAL DOMAIN

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1. Introduction

Over the past forty years many examples in commutative algebra have been constructed using the following principle: Let k be a field, let $S = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ be a localized polynomial ring over k, and let $\mathbf{a}$ be an ideal in the completion $\widehat{S}$ of S such that the associated primes of $\mathbf{a}$ are in the generic formal fiber of S; that is, $\mathbf{p} \cap S = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(\widehat{S}/\mathbf{a})$. Then S embeds in $\widehat{S}/\mathbf{a}$, the fraction field $Q(S)$ of S embeds in the fraction ring of $\widehat{S}/\mathbf{a}$, and for certain choices of $\mathbf{a}$, the intersection $D = Q(S) \cap (\widehat{S}/\mathbf{a})$ is a local Noetherian domain with completion $\widehat{D} = \widehat{S}/\mathbf{a}$.

Examples constructed by this method include Nagata's first examples of non-excellent rings [N], Ogoma's celebrated counterexample to Nagata's catenary conjecture [O1], [O2], examples of Rotthaus and Brodmann [R1], [R2], [BR1], [BR2], and examples of Nishimura and Weston [Ni], [W]. In fact all examples we know of local Noetherian reduced rings which contain and are of finite transcendence degree over a coefficient field may be realized using this principle.[1]

The key to these examples is usually the behavior of the formal fibers of the domain D. A major problem in this setting is to identify and classify ideals in the formal fiber of S according to the properties of the intersection domain $D = Q(S) \cap (\widehat{S}/\mathbf{a})$. The goal of this paper is to study the significance of the choice of the ideal $\mathbf{a}$ in this construction.

In many of the examples mentioned above, the expression $D = Q(S) \cap (\widehat{S}/\mathbf{a})$ may be interpreted so that D is an intersection of the completion of a local Noetherian domain R with a subfield. In this paper we consider this latter form. More precisely we use the following setting throughout this paper.

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[1] We conjecture that all local Noetherian reduced rings D which contain a coefficient field k and which are of finite transcendence degree over k relate to an ideal $\mathbf{a}$ in the generic formal fiber of the localization of a polynomial ring $S = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, in such a way that D is a direct limit of étale extensions of such an intersection ring $Q(S) \cap (\widehat{S}/\mathbf{a})$ as above.