OBSTRUCTIONS TO IMPOSING DIFFERENTIABLE STRUCTURES

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A classical problem of differential topology is the following, which has been called the *regularity problem*: Given a topological n-manifold M, does M possess a differentiable structure? Recently, M. Kervaire has found an example in dimension 10 which shows the answer is "not always" [6]. On the other hand, the affirmative answer is known to hold for $n \leqq 3$, at least if M is nonbounded (i.e., if Bd M is empty); one combines the triangulation theorem of E. E. Moise [8] with work of S. S. Cairns [2].

To obtain further results, one usually adds more hypotheses to the problem, requiring M to possess a triangulation which is nice, in some sense. Specifically, one requires that the triangulation make M into a *combinatorial manifold*, in which the closed star of every vertex has a subdivision which is isomorphic with a rectilinear subdivision of the standard n-simplex. (It follows readily that some subdivision of this triangulation is a *Brouwer* triangulation [1]. This means that the closed star of every vertex is imbeddable in R^n by a homeomorphism which is linear on each simplex, and if the vertex lies on Bd M, the image of the open star is an open subset of the half-space $x_1 \geqq 0$.) This extra hypothesis is reasonable, since if M has a differentiable structure, it also has a compatible triangulation [12], [14], which is automatically combinatorial (see 8.4 of [12]).

In this paper, we apply our previously developed techniques [11] to this problem. Roughly, our approach is to assume a Brouwer triangulation of M, and take the imbeddings $l_v : \mathrm{Cl}(\mathrm{St}\, v) \to R^n$ as a first try at coordinate systems covering M. These do not overlap differentiably, but we attempt to "smooth them out" so that they will. Obstructions to this smoothing are encountered, which appear in $\mathcal{H}_{m-1}(M, \mathrm{Bd}\, M; \Gamma^{n-m})$. Here $\mathcal{H}$ denotes infinite homology, with twisted coefficients in the nonorientable case. Γ^{n-m} is the group of orientation-preserving diffeomorphisms of S^{n-m-1}, modulo those extendable to the ball B^{n-m}. In special cases (e.g., if M is contractible) all these homology groups vanish, and our techniques suffice to construct a differentiable structure on M. A list of such cases appears in 2.12.

Once one has strengthened the hypotheses, one may also wish to strengthen the conclusion of the problem, and require that the differentiable structure obtained should be compatible with the given triangulation, or some subdivision of it. The differentiable structures we construct do not have this property. For example, they may possess *conical points*: If we imbed a

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