LIE ALGEBRAS WITH SUBALGEBRAS OF CO-DIMENSION ONE

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In the study of Lie groups of dimension n acting on a space and having a 1-dimensional orbit—and in particular in the study of the boundary of certain topological semigroups on manifolds with boundary—the following problem arises: If G is an n-dimensional connected Lie group and H a closed $(n - 1)$-dimensional subgroup, is there a one-parameter group E such that $G = HE$ and no conjugate of E is contained in H? If so, "how many" such one-parameter groups exist? We shall give a complete answer to this question. In order to do so we classify the n-dimensional real Lie algebras possessing a subalgebra of dimension $n - 1$ (Theorem I). Thus we establish the fact that the Lie algebra of G contains at least $n - 1$ linearly independent vectors such that no conjugate of a one-parameter group generated by one of these is ever contained in H; in many cases there are even n linearly independent vectors with this property.

In order to make the proof fairly self contained we first deal with simple Lie algebras; the results so obtained may also be produced by a close inspection of the classification of simple Lie algebras.

LEMMA 1. *Let $\mathfrak{G}$ be a compact simple Lie algebra over an ordered field. Suppose that* $\dim \mathfrak{G} = n$ *and that $\mathfrak{H}$ is a subalgebra of dimension $n - d$. Then* $2n \leq d(d + 1)$.

Proof. Since usually the term of a compact Lie algebra is applied to real Lie algebras we first remark, that under a compact Lie algebra over an ordered field we understand a Lie algebra whose Killing form is negative definite. Now we let $\mathfrak{B}$ be an orthogonal complement of $\mathfrak{H}$ in $\mathfrak{G}$ with respect to the Killing form. Then $\dim \mathfrak{B} = d$. Preserving the Killing form on $\mathfrak{G}$ under the adjoint action, $\mathfrak{H}$ is represented in the Lie algebra of the orthogonal group $O(\mathfrak{B})$ on $\mathfrak{B}$. Let $\mathfrak{K}$ be the kernel of this representation. Then

$$[\mathfrak{K}, \mathfrak{G}] = [\mathfrak{K}, \mathfrak{B}] + [\mathfrak{K}, \mathfrak{H}]$$

which is in $\mathfrak{K}$ because the first summand vanishes and $\mathfrak{K}$ is an ideal in $\mathfrak{H}$. This shows that $\mathfrak{K}$ is an ideal of $\mathfrak{G}$. Since $\mathfrak{G}$ is simple we have $\mathfrak{K} = 0$; so $n - d = \dim \mathfrak{H} \leq \dim O(\mathfrak{B}) = d(d - 1)/2$. Consequently $2n \leq d(d - 1)$.

It may be remarked that equality holds if $\mathfrak{G}$ is the Lie algebra of $SO(m)$ and $\mathfrak{H}$ is the Lie algebra of the subgroup $SO(m - 1)$.

LEMMA 2. *Let $\mathfrak{G}$ be a real simple n-dimensional Lie algebra and $\mathfrak{H}$ a subalgebra of dimension $n - 1$. Then* $\dim \mathfrak{G} = 3$ *and* $\mathfrak{G} \cong sl(2)$, *the Lie algebra of* $Sl(2)$.

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