# REMARKS ON NONLINEAR FUNCTIONAL EQUATIONS, III

BY

FELIX E. BROWDER[1]

## Introduction

In two preceding papers under the same title [2], [3], the writer has studied nonlinear functional equations in reflexive complex Banach spaces involving operators $T$ mapping such a Banach space $X$ into its dual $X^*$ and satisfying inequalities of the type

$$|(Tu - Tv, u - v)| \geq k(u, v)h(\| u - v \|) \tag{1}$$

where $(w, v)$ denotes the pairing between $w$ of $X^*$ and $v$ of $X$.

A representative result of this type is Theorem 1 of [2] which asserts that if $T$ is demicontinuous and saeisfies the two conditions:

(i) There exists a real function $c(r)$ on $R^1$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $u$ of $X$,

$$|(Tu, u)| \geq c(\| u \|)\| u \|; \tag{2}$$

(ii) For each $N > 0$, there exists a continuous strictly increasing real function $k_N(r)$ on $R^1$ with $k_N(0) = 0$ such that for $\| u \| \leq N$, $\| v \| \leq N$

$$|(Tu - Tv, u - v)| \geq k_N(\| u - v \|)\| u - v \|; \tag{3}$$

then $T$ maps $X$ onto $X^*$.

This theorem is an extension and generalization of a theorem of Zarantonello [7] which asserts that if $T$ is a continuous map of a Hilbert space $H$ into $H$ which carries bounded sets into bounded sets and such that for a suitable constant $c > 0$

$$|(Tu - Tv, u - v)| \geq c \| u - v \|^2$$

then $T$ maps $H$ onto $H$.

Further extensions were given by the writer in [3] in which on the one hand the inequality (3) of (ii) was modified to

$$|(Tu - Tv, u - v)| \geq k_N(\| u - v \|) - |(C_N u - C_N v, u - v)| \tag{4}$$

where for each $N > 0$, $C_N$ is some completely continuous map of $X$ into $X^*$, and on the other, the demicontinuity of $T$ was replaced by the condition that $T = L + G$ where $G$ is demicontinuous and maps bounded sets of $X$ into bounded sets of $X^*$ while $L$ is a closed densely defined linear map of $X$ into $X^*$ such that its adjoint $L^*$ is the closure of its restriction to $D(L) \cap D(L^*)$.

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