SOME PROPERTIES OF A SINGULAR DIFFERENTIAL OPERATOR

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1. Introduction

Let X denote the B-space of all bounded and uniformly continuous functions from the real line E_1 into the set C of all complex numbers. Suppose p is a function from E_1 into E_1 which has uniformly bounded difference quotients, and let $F(p)$ denote the class of all functions from E_1 into C which are differentiable at every non-zero of p. For each x in $F(p)$, let x^* be defined on E_1 by $x^*(s) = x'(s)$ if $p(s) \neq 0$, $x^*(s) = 0$ if $p(s) = 0$. Let D denote the linear subspace of X consisting of all x in X such that x is in $F(p)$ and px^* is in X, and let A denote the linear transformation from D into X defined by $Ax = px^*$. Let D^2 denote the linear subspace of X consisting of all x in D for which Ax is in D, and let A^2 denote the linear transformation from D^2 into X defined by $A^2x = A(Ax)$. Various properties are developed for the transformations $A, A^2, A + Q$, and $A^2 + PA + Q$, where P and Q denote bounded linear transformations from X into X, and the results have applications to partial and ordinary differential equations. The results all carry over if X is taken to be the B-space of all bounded and uniformly continuous complex-valued functions defined on an interval $[a, b]$ if $p(a) = p(b) = 0$, $[a, \infty)$ if $p(a) = 0$, or $(-\infty, b]$ if $p(b) = 0$.

Most of the main results require that p be bounded and are obtained by first establishing the fact that A is the infinitesimal generator of a strongly continuous group $[T(t), -\infty < t < \infty]$ of bounded operators in X and giving a simple formula for $T(t)$. This yields the fact that A^2 is the infinitesimal generator of a strongly continuous semi-group $[V(t), 0 \leq t < \infty]$ of bounded operators in X and a formula for $V(t)$. The theory of semi-groups of operators is discussed by Dunford and Schwartz in [2] and more completely by Hille and Phillips in [4]. In this paper, as in [2], the term "strongly continuous" means belonging to the class $(C, 0)$ of [4]. In some of the applications to differential equations, advanced calculus methods are used to sharpen the results beyond what the semi-group theory alone would yield.

Glazman [3], Stone [5], and Weyl [6] have treated similar singular differential operators. They considered the differential operators as operators in a Hilbert space of Lebesgue square-integrable functions and allowed singularites to occur only at the end points of the domain of the functions in the space. In this paper, singularities (even intervals of them) are allowed to occur within the interval on which the functions considered are defined, but the results are not as complete.

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