TRIGONOMETRIC POLYNOMIALS IN PRIME NUMBER THEORY[1]

BY
STEVEN H. FRENCH

1. Introduction

Let C_n be the set of n^{th} degree cosine polynomials g such that if

$$g(\phi) = a_0 + a_1 \cos \phi + \cdots + a_n \cos n\phi,$$

then $g(\phi) \geqq 0$ for all real ϕ, $a_1 > a_0 > 0$, and $a_k \geqq 0$ for $k = 2, 3, \cdots, n$. The estimates for the errors in approximate formulas obtained for various functions of prime numbers depend on the following two quotients formed from the coefficients of members of C_n :

$$R = R(g) = \frac{a_1 + a_2 + \cdots + a_n}{2(\sqrt{a_1} - \sqrt{a_0})^2},$$

$$S = S(g) = \frac{a_0 + a_1 + \cdots + a_n}{a_1 - a_0}.$$

Following standard notation, we denote by $\pi(x)$ the number of primes less than or equal to x. For $\pi(x)$, Landau [5, vol. 1, pp. 242–251] established the validity of the following approximation for all $\lambda > S + 2$:

$$\pi(x) = \int_2^x \frac{dy}{\log y} + O(xe^{-(\log x)^{1/\lambda}}).$$

By more sophisticated arguments Landau [5, vol. 1, pp. 321–333] also showed that for any $\rho > R$ we have

$$\pi(x) = \int_2^x \frac{dy}{\log y} + O(x (\log x)^{-1/2} e^{-\sqrt{(\log x)/\rho}}).$$

The last estimate for the error for large x depends on the following result concerning the zeros of the Riemann zeta-function $\zeta(s)$. If $a > R$, there exists a positive number γ_0 depending on a such that if $\beta + i\gamma$ is a zero of $\zeta(s)$ with $\gamma \geqq \gamma_0$, then $\beta < 1 - 1/(a \log \gamma)$. From these results it can be seen that the estimates for the error are decreased if S and R are made smaller.

The problem involving S has been treated by Landau, Tschakaloff and van der Waerden. Denoting the g.l.b. of $S(g)$ for $g \in C_n$ by P_n, the best results can be summarized as follows. Tschakaloff [8] proved that $P_2 = 7, P_3 = P_4 = P_5 = 6, P_6 = 5.92983 \cdots$, and $P_7 = P_8 = P_9 = 5.90529 \cdots$; he gave another

Received August 14, 1964; received in revised form January 14, 1965.

[1] This work was done at the Ordnance Research Laboratory of The Pennsylvania State University, University Park, Pennsylvania. The author's present address is TRACOR, Inc., 627 Lofstrand Lane, Rockville, Maryland. The thesis [3], which contains the results presented here, was directed by Professor Lowell Schoenfeld.