THE SPACE OF HOMEOMORPHISMS OF A DISC WITH n HOLES

BY

H. R. MORTON

In [3] M-E. Hamstrom and E. Dyer showed that the identity component of the space of homeomorphisms of an annulus onto itself, keeping the boundary pointwise fixed, is contractible. In [4] Hamstrom showed that, denoting by H_n the identity component of the space of homeomorphisms of a disc with n holes onto itself, keeping the boundary pointwise fixed, we have $\pi_i(H_n) = 0$ for all $i > 0$.

This paper shows that H_n is contractible for all n.

Remark. On all function spaces we shall use the compact-open topology, and refer the reader to Hu [5] for results. In particular, for the space X^Y, where X and Y are both compact metric, the induced metric topology agrees with the compact-open topology, [5, p. 102].

The theorem is proved in two steps, which are summarized here.

Define H_n^* to be the identity component of the space of homeomorphisms of a disc with n holes onto itself, keeping the boundary, and also one interior point a, pointwise fixed.

LEMMA 1. *H_{n-1} deformation retracts onto H_{n-1}^*.*

(A sketch proof follows. The details are given later.)

To every homeomorphism h in H_{n-1} we assign continuously a point $s(h)$ in in the interior of the universal cover of the manifold which lies above the point $h(a)$.

To every point of the interior of the universal cover we assign a canonically defined path to a chosen base point lying above a, and also a canonical isotopy of the manifold, keeping the boundary fixed. This isotopy starts at the identity, and makes the projection of the point follow the projection of the canonical path to the point a. By following a homeomorphism h with this isotopy for the point $s(h)$ we get a canonical isotopy of h to a homeomorphism h^*, which keeps a fixed. This h^* lies in H_{n-1}^*.

LEMMA 2. *H_{n-1}^* is homotopy equivalent to H_n.*

(The details of this proof are given later.)

We define an inclusion $i : H_n \to H_{n-1}^*$ by filling in one hole of the disc with n holes and extending $h \in H_n$ over this by the identity. Taking a as the centre of this filled-in disc, the extended map is in H_{n-1}^*.

The reverse map, $r : H_{n-1}^* \to H_n$ is constructed using a technique of H.

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