CONVEXITY OF POLYHEDRA

BY
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Introduction

The intuitive feeling persists that the convexity of a polyhedron should be determined by the nature of the vertex set of the polyhedron. A few minutes with paper and pencil, however, convince the newcomer to convexity that the precise relationship between the vertex set and convexity is not an obvious one. This paper investigates that relationship. In Section II we demonstrate that a finite geometric simplicial n-complex (polyhedron) is convex if and only if it fulfills a certain vertex condition similar to convexity, is contained in an n-hyperplane and is either starlike or $(n - 1)$-connected. In Section III these results are applied to the problem of subdividing a complex in such a way as to make the star at each vertex convex. Some of the complexes in Section III are not simplicial, but, in the absence of explicit comment to the contrary, it should be assumed that all complexes are simplicial.

I. Definitions

If S and T are sets in E^n (real n-space), $S \circ T$ denotes the union of all closed line segments st with $s \in S$ and $t \in T$. It is easy to see that a set X in E^n is convex if and only if $S \subset X$ and $T \subset X$ imply $S \circ T \subset X$. If K is a complex, $| K |$ will denote the point set occupied by K although, where the distinction is not important, we may refer to the complex K itself (as opposed to the set $| K |$) as convex. A set T in E^n is n-connected if and only if every map $f : S^k \to T$ $(0 \leqq k \leqq n)$ is homotopic to a constant map $g : S^k \to T$, where S^k denotes the k-sphere. A set T in E^n is starlike if and only if it contains a point v such that $s \in T$ implies the segment vs is contained in T. A polyhedron K is *n-vertex-convex* (n-vc) if and only if the simplex spanned by any $n + 1$ of its vertices is contained in $| K |$. All discussions take place in E^n.

II. Vertex convexity and convexity

If K is the n-complex obtained by joining the barycenter of the n-simplex to each of its vertices, let K' be the complex obtained by removing one of the n-simplexes of K and the $(n - 1)$-face of this simplex which lies on the boundary of the original simplex. The set $| K' |$ is not convex, but it is starlike, $(n - 1)$-connected and lies in an n-hyperplane. The complex K' however is not $(n - 1)$-vc although it is $(n - 2)$-vc. The key to assuring convexity is $(n - 1)$-vertex convexity.

Theorem 1. *If K is an n-polyhedron, $(n \geqq 2)$, then $| K |$ is convex if and*

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