ON THE CONCORDANCE OF LOCAL RINGS AND UNIQUE FACTORIZATION

BY

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Introduction

Every ring in this paper is commutative with an identity and a subring must always have the same identity as the containing ring. A ring A which has only one maximal ideal m is a quasi-local ring and is denoted by (A, m); a quasi-local ring (A, m) is a local ring if $\bigcap_{n=1}^{\infty} m^n = (0)$. By the natural topology of a quasi-local ring (A, m), we mean the m-adic topology of A.

The purpose of this paper is to investigate unique factorization in the direct limits of directed systems $(R_p, f_p^q)_{p \in I}$ of unique factorization domains (or UFD's) R_p relative to any directed set I. In particular, we concentrate on the case that all the UFD's are local rings (R_p, m_p) and every homomorphism f_p^q, $q \geq p$, is a local homomorphism; namely $f_p^q(m_p) \subseteq m_q$ for $q \geq p$. Let R be the direct limit of such a directed system of UF local domains R_p, and g_p the natural map of R_p into R for each $p \in I$. We prove that R is a UF local domain, if $g_p(R_p)$ for every $p \in I$ is a local ring having R as a concordant extension; that is, $g_p(R_p)$ is a topological subspace of R for the natural topologies. Applying the result and the fact that every regular (Noetherian) local ring is a UFD, we also prove that if $(E_p, f_p^q)_{p \in I}$ is a directed system of regular local rings (E_p, m_p) such that each homomorphism f_p^q maps a minimal basis of m_p into that of m_q for $q \geq p$, then the direct limit is a UF local domain.

1. Concordant extensions of local rings

Throughout this paper we shall denote by N the set of all positive integers.

DEFINITION. Let R_p and R_q be two quasi-local rings such that $R_p \subseteq R_q$; then R_q is said to be a *concordant extension* of R_p, if R_p is a subspace of R_q for the natural topologies.

From the definition we know that (R_q, m_q) is a concordant extension of (R_p, m_p) if and only if for any two integers h and k in N, we can find r and s in N such that

$$m_p^r \subseteq m_q^h \cap R_p \quad \text{and} \quad m_q^s \cap R_p \subseteq m_p^k .$$

Thus $R_p \subseteq R_q$ and $m_q \cap R_p = m_p$, i.e., R_q dominates R_p, is a necessary condition for R_q to be a concordant extension of R_p (cf. [2]).

For any directed set I let $(R_p)_{p \in I}$ be a family of local rings (R_p, m_p) such that R_q dominates R_p if $q \geq p$. Then the set $(R_p)_{p \in I}$ is a directed system of local rings under set inclusion; the direct limit $R = \lim_{\rightarrow} R_p = U_{p \in I} R_p$ is a

Received April 4, 1966.