ON UNIQUE FACTORIZATION IN ALGEBRAIC FUNCTION FIELDS

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1. Introduction

Let K be a field of algebraic functions of one variable over an algebraically closed field k and let R be an integrally closed sub-domain of K, properly containing k, which is contained in all but a finite number of valuation rings of K/k. Cunnea [3, Corollary 4.2] has proved that R is a unique factorization domain if and only if K has genus 0. The present writer [1][1] has discussed the question of the existence of a euclidean algorithm in a ring which is essentially like R and, in particular, has proved that R is euclidean if K has genus 0. As usual, the existence of a euclidean algorithm in R implies that factorization is unique. In the light of this and of Cunnea's results the following is perhaps of interest.

THEOREM. *Let K be a field of algebraic functions of one variable over an infinite field k and let R be an integrally closed sub-domain of K, properly containing k, which has no poles outside a finite set $S = \{\mathfrak{P}_1, \cdots, \mathfrak{P}_s\}$ of places of K/k. Then R is euclidean if and only if*

$$g + d_S = 1, \tag{1}$$

where g is the genus of K and d_S is the greatest common divisor of the degrees of the places in S.

We recall the essential results of [1] and deduce the sufficiency part of the theorem in §2. In §3 we prove a lemma on linear spaces and the proof of the theorem is concluded in §4. The case of finite k is mentioned in §5.

2. Euclid's algorithm in function fields

Let $\mathfrak{b}$ be a divisor of K based on the set S and let $\mathfrak{L}(\mathfrak{b}, S)$ denote the set

$$\mathfrak{L}(\mathfrak{b}, S) = \{\beta \in K : \nu_{\mathfrak{P}_i}(\beta) \geq \nu_{\mathfrak{P}_i}(\mathfrak{b}), \mathfrak{P}_i \in S\}, \tag{2}$$

where $\nu_{\mathfrak{P}_i}$ denotes the order function at $\mathfrak{P}_i$. By a straightforward adaptation of the argument in [1], it follows that R is a euclidean domain if and only if

$$K = \bigcup (\mathfrak{L}(\mathfrak{b}, S) + R), \tag{3}$$

where the union is taken over all divisors $\mathfrak{b}$ based on S such that $\deg(\mathfrak{b}) \geq 1$. Moreover

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[1] In [1], k was a finite field; the extension to an infinite field presents no difficulty. Section 7 of [1] is fallacious, but is not relevant to the present paper; see Corrigendum and Addendum to appear in J. London Math. Soc.