WIENER'S TESTS FOR ATOMIC MARKOV CHAINS[1]

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1. Introduction

In a paper of Itô and McKean [1] a criterion, known as Wiener's test, is obtained for deciding whether a set of states in the simple d-dimensional random walk will be visited finitely often with probability 0 or 1. Lamperti has shown [2] that a similar test is valid for a class of Markov chains which includes all d-dimensional random walks with zero means and finite second moments.

In this paper we obtain necessary and sufficient conditions for the existence of Wiener-type tests in arbitrary discrete parameter Markov chains with stationary transition probabilities.

We follow closely the terminology of Chung [3]. The state space I is taken as the set of positive integers. $P = (p_{ij})$ $(i, j \in I)$ is the matrix of (one-step) transition probabilities, its n^{th} power $P^n = (p_{ij}^{(n)})$ is then the matrix of n-step transition probabilities. Ω is the set of infinite sequences $\omega = (i_0, i_1, \cdots)$ with $i_t \in I$ $(t \geq 0)$. The probability measure $\Pr(\cdot)$ on Ω is fully determined once the initial probability distribution of states i_0 is known, $p_j = \Pr(i_0 = j)$. Usually we will only be interested in conditional probabilities $\Pr(\cdot \mid i_0 = j)$ where the initial state is fixed. The successive states of a sample path are labelled $x_0, x_1, \cdots$ and we say that the Markov chain is in state i at time n if $x_n = i$. For any element $\omega = (i_0, i_1, \cdots)$ of Ω we write $x_n(\omega) = i_n$.

We define the Green's function of the chain

$$G_{ij} = \textstyle\sum_{n=0}^{\infty} p_{ij}^{(n)} \tag{1.1}$$

where for convenience we put $p_{ij}^{(0)} = \delta_{ij}$. If G_{ii} is finite we say that the state i is transient, otherwise it is recurrent. The set R of recurrent states of I can be divided into disjoint recurrent classes $R_u = \{i : G_{ui} > 0\}$ corresponding to certain recurrent states u.

For any set of states A we define the functions

$$f_m(i, A) = \Pr(x_n(\omega) \in A \quad \text{for at least} \quad m \quad \text{values of} \quad n \mid x_0(\omega) = i) \tag{1.2}$$

$$\begin{aligned} h(i, A) &= f_\infty(i, A) \\ &= \lim_{m\to\infty} f_m(i, A) = \Pr(x_n(\omega) \in A \quad \text{infinitely often} \quad \mid x_0(\omega) = i) \end{aligned} \tag{1.3}$$

$$\begin{aligned} e_m(i, A) &= f_m(i, A) - f_{m+1}(i, A) \\ &= \Pr(x_n(\omega) \in A \quad \text{exactly} \quad m \quad \text{times} \quad \mid x_0(\omega) = i). \end{aligned} \tag{1.4}$$

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