ALGEBRAIC MODELS FOR MEASURES

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1. Introduction

The purpose of this paper is to study probability measure spaces (X, Σ, μ) by means of algebraic models (Γ, φ) consisting of an abelian group Γ and a function of positive type φ on Γ (see Definitions 2 and 3). Algebraic models determine uniquely the measures, in the sense that two measures are essentially equal, or conjugate (see Definition 1) if and only if they possess isomorphic algebraic models (Theorem 2). Every algebraic measure system (Γ, φ) is an algebraic model for a certain measure (Theorem 3). In particular, we obtain a new reduction of a measure μ on an abstract set, to a regular Borel measure μ' on an abelian compact group (Theorem 4), and we give conditions in order that μ' should be a Haar measure (Theorem 5).

2. Conjugate measures

Let (X, Σ, μ) be a probability measure space. We denote by $\Gamma(\mu)$ the set of the (equivalence classes of) functions $f \in L^{\infty}(\mu)$ with $|f| \equiv 1$. Then $\Gamma(\mu)$ is a *multiplicative group* with the complex conjugate $\bar{f}$ as inverse of a function $f \in \Gamma(\mu)$. If we identify the circle group C with the constant functions of $\Gamma(\mu)$, we have $C \subset \Gamma(\mu)$.

Remark. Using the existence of a lifting (see [4]) we can consider that $\Gamma(\mu)$ is a group of μ-measurable functions $f : T \to E$ with $|f| \equiv 1$, such that $f, g \in \Gamma(\mu)$ and $f(x) = g(x)$ (μ-almost everywhere) imply $f(x) = g(x)$ for every $x \in X$.

We define the complex function φ_μ on $\Gamma(\mu)$ by

$$\varphi_\mu(f) = \int f \, d\mu \quad \text{for} \quad f \in \Gamma(\mu).$$

PROPOSITION 1. *φ_μ is a function of positive type on $\Gamma(\mu)$ and*

$$\varphi_\mu(f) = 1 \quad \textit{if and only if} \quad f = 1.$$

In fact, for every family $(f_i)_{1 \le i \le n}$ of functions of $\Gamma(\mu)$ and for every family $(\alpha_i)_{1 \le i \le n}$ of complex numbers we have

$$\sum_{i,j} \alpha_i \bar{\alpha}_j \varphi_\mu(f_i f_j^{-1}) = \sum_{i,j} \alpha_i \bar{\alpha}_j \int f_i \bar{f}_j \, d\mu = \int |\sum_k \alpha_k f_k|^2 \, d\mu \ge 0;$$

therefore φ_μ is of positive type.

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