THE FREE PRODUCT OF ALGEBRAS[1]

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Introduction

Let A and B be differential graded augmented algebras over a commutative ring K. Their free product $A * B$ is always defined; $A * B$ is a differential graded augmented K-algebra which together with canonical injections

$$A \xrightarrow{\iota_A} A * B \xleftarrow{\iota_B} B$$

forms a universal diagram in this category. In connection with certain topological questions, Berstein [1] first studied the free product of algebras and its homology; he showed for example that the homology of the loop space of $X_1 \vee X_2$ (where X_i are spaces with "nice" base point) is the free product $H(\Omega X_1) * H(\Omega X_2)$. We shall study the free product and its homology from a somewhat different viewpoint.

The first section is devoted to the definition and basic properties of the free product, including a consideration of Hopf algebras. Some of this material appears in Berstein [1], but is stated here for convenience since our notation is different and our definitions are somewhat more general (Berstein considers only positively graded connected K-algebras).

Palermo [10] and the author [5], [6] have studied the relationship between the vaious homologies $H(A)$, $H(B)$, and $H(A \otimes B)$. The chief purpose of this paper is to extend these investigations to $H(A * B)$. In particular since $A * B$ is defined in terms of the tensor product it seems natural to ask whether or not $H(A \otimes B)$ completely determines $H(A * B)$. Examples in Section 2 show that the answer is negative; furthermore neither does $H(A * B)$ determine $H(A \otimes B)$. For $K = Z$ and A, B torsion-free, it is known that the algebras $H(A)$ and $H(B)$ do not determine the algebra $H(A \otimes B)$; but $H(A \otimes B)$ is completely determined by the homology spectra of A and B (cf. Palermo [10], and [5]). The analogues of these facts are presented in Section 3: $H(A)$ and $H(B)$ are not sufficient to determine $H(A * B)$ (Example 3.4), but the algebra $H(A * B)$ is completely determined by the homology spectra of A and B (Theorem 3.3).

In the final sections, the work of Dold and Puppe [4] is used to develop a theory of derived functors for the nonadditive functor $A * B$. Not surprisingly these derived functors turn out to be closely related to the ordinary derived functors of the multiple tensor product (c.f [6]). Using these results we are able to state a "Künneth theorem" which relates the (additive) struc-

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