EIGENVALUES AND BOUNDARY SPECTRA

C. R. PUTNAM[1]

1. Introduction

Let T be a bounded operator on a Hilbert space of elements x with $\| T \| = \sup \| T \|$, where $\| x \| = 1$, and let $\operatorname{sp}(T)$ denote the spectrum of T. It is well known that if $\lambda \notin \operatorname{sp}(T)$ then

$$\|(T - \lambda I)^{-1}\| \geqq 1/d_T(\lambda),$$

where $d_T(\lambda)$ denotes the distance from λ to the (compact) set $\operatorname{sp}(T)$. This paper will deal with bounded operators for which the equality sign holds, so that

$$(1) \qquad \|(T - \lambda I)^{-1}\| = 1/d_T(\lambda), \qquad \lambda \notin \operatorname{sp}(T).$$

The condition (1) is known to be valid for normal operators ($TT^* = T^*T$) as well as for semi-normal ones ($TT^* - T^*T$ semi-definite); cf. Stampfli [5]. Also, as is noted there, if T satisfies (1) and if $\operatorname{sp}(T)$ is real then T is self-adjoint (Nieminen [3]), and if T satisfies (1) and $\operatorname{sp}(T)$ lies on the unit circle then T is unitary (Donoghue [1]). Further, Stampfli has shown that if T satisfies (1) and if λ is an isolated point of $\operatorname{sp}(T)$ then necessarily λ is in the point spectrum of T, so that $Tx = \lambda x$ holds for some $x \neq 0$. In addition, $T^*x = \bar{\lambda}x$, so that the eigenvectors of T belonging to λ determine a reducing space on which T is normal. Thus, if T satisfies (1) on a finite-dimensional Hilbert space, then T must be normal.

Concerning other implications of (1) and of related growth restrictions on the resolvent operator see, in addition to the references cited above, also Meng [2], Orland [4].

It will be convenient to have the following

DEFINITION. Let λ belong to the point spectrum of an operator T and suppose that the eigenvectors of T belonging to λ determine a reducing space of T on which T is normal. Then λ will be called a normal eigenvalue of T.

Thus, by Stampfli's result, isolated points of the spectrum of an operator satisfying (1) must be normal eigenvalues. It will be shown below that an analogous assertion holds for certain other eigenvalues lying in the boundary of the spectrum of an operator T satisfying (1).

2. The main theorem

THEOREM. *Let T satisfy* (1) *and suppose that λ_0 is in the point spectrum of T and also in the boundary of* $\operatorname{sp}(T)$. *Suppose that there exist $\lambda_n \notin$* $\operatorname{sp}(T)$ *for*

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