BEHAVIOR OF LINEAR FORMS ON EXTREME POINTS[1]

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Introduction

Suppose that L is a locally convex topological linear space, L^* is its conjugate space, K is a compact convex subset of L, and E_K is the set of all extreme points of K. By the Krein-Milman Theorem [9], [3, pp. 83–84], $K = \text{cl con } E_K$ and hence $\sup fE_K = \sup fK$ for each $f \in L^*$; moreover, the K-supremum of f is actually attained on E_K. If E_K is finite (that is, if K is a finite-dimensional convex polytope) then

(I) *for each* $f \in L^*$ *the number* $\sup fE_K$ *is an isolated point of the set* fE_K.

Here we are concerned with the restrictions which condition (I) imposes on the topological structure of E_K. The following results are established.

(a) *When L is finite-dimensional, condition* (I) *is equivalent to the finiteness of* E_K

(b) *When L is a Banach space,* (I) *implies the isolated points of* E_K *are dense in* E_K. *Conversely, for each infinite-dimensional Banach space L and each compact metric space Q in which the isolated points are dense, there is a compact convex set K in L such that* (I) *holds and* E_K *is homeomorphic with Q.*

(c) *In general,* (I) *does not imply E has isolated points. Indeed, for each totally disconnected compact Hausdorff space Q there is a locally convex space L and a compact convex set K in L such that* (I) *holds and* E_K *is homeomorphic with Q.*

These results are for compact convex sets, but some related results are obtained for locally compact closed convex sets containing no line.

Two lemmas

The lemmas of this section will not be used in proving (a), (c), or the first part of (b). They will be used for the second part of (b) and for a related finite-dimensional result.

For subsets X and Y of a metric space with distance function ρ, let

$$\delta(X, Y) = \inf \{\rho(x, y) : x \in X, y \in Y\}.$$

For a point x of the space let $\delta(x, Y) = \delta(\{x\}, Y)$. The following result may well be known, but lacking a specific reference we include a proof.

TOPOLOGICAL LEMMA. *For* $r = 1, 2$, *suppose that* A_r *is the set of all accumulation points of a compact metric space* Q_r *and that the set* $Q_r \sim A_r$ *of all*

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