AN EXTENSION OF RADON'S THEOREM[1]

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1. introduction

Let "m-set" mean a set of m points in the d-dimensional space R^d. An m-set is said to be (r, k)-divisible if it can be partitioned into r pair-wise disjoint subsets in such a way that the intersection of the convex hulls of these r subsets is at least k-dimensional. (We always assume $0 \leq k \leq d$. The empty set is (-1)-dimensional, while 0-dimensional sets are non-empty.)

A classic theorem of J. Radon [5] asserts that each $(d + 2)$-set is $(2, 0)$-divisible. The first generalization for $r > 2$ was given by R. Rado [4]. B. Birch [1] conjectured (and proved for $d = 2$) that each $((d + 1)(r - 1) + 1)$-set is $(r, 0)$-divisible, while H. Tverberg [6] established this conjecture for all values of d. It is clear that if $k > 0$ then other conditions on a given m-set S besides a lower bound on its cardinality are necessary if S is to be (r, k)-divisible. For example, if all the points of S were on a line in R^d, no subset would have a convex hull of dimension greater than one. The purpose of this paper is to consider various types of independence that may be imposed upon an m-set to insure (r, k)-divisibility (Section 3) and to prove the following theorem which extends the results mentioned above.

THEOREM 1. *Each* $[(d + 1)(r - 1) + k + 1]$*-set of strongly independent points in* R^d *is* (r, k)*-divisible.*

A set S in R^d is said to be *strongly independent* provided that each finite family $\{S_1, \cdots, S_r\}$ of pair-wise disjoint subsets of S has the following property:

$$\text{If } d_i = (\text{card } S_i) - 1 \leq d, \text{ then}$$

$$\dim \left(\bigcap_{i=1}^{r} \text{aff } S_i\right) = \max \left(-1, d - \sum_{i=1}^{r} (d - d_i)\right). \tag{1}$$

(Condition (1) may be thought of as follows: Since $(d - d_i)$ is just the deficiency of aff S_i when S_i is in general position, condition (1) implies the general position of S and its subsets. Thus the right side of the equation is essentially the dimension of the space reduced by the deficiencies of the flats aff S_i. This keeps the flats aff S_i from forming "pencils of lines", "books of planes", etc.)

We will let lin S, aff S, card S, and conv S denote respectively the linear

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