ON LIMIT-PRESERVING FUNCTORS

BY

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Following Lambek [2] we shall use the suggestive term "infimum" for the generalized inverse limit of Kan. "Supremum" is defined dually. In [1], the infimum (supremum) is known as a "left root" ("right root"). The terms "inf-complete" and "inf-preserving" are used in the obvious way.

If $\mathfrak{A}$ is a small category then $[\mathfrak{A}, \text{Ens}]$ shall denote the category of all (covariant) functors from $\mathfrak{A}$ to the category Ens of sets. $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$ shall be the full subcategory of inf-preserving functors.

The theorem below answers an open question raised in the introduction to [2]. As Lambek points out this result implies that $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$ is sup-complete and can be regarded as a nicely behaved completion of $\mathfrak{A}^{\circ}$, the dual or opposite category of $\mathfrak{A}$.

THEOREM. *Let* $\mathfrak{A}$ *be a small category. Then* $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$ *is a reflective subcategory of* $[\mathfrak{A}, \text{Ens}]$.

Notation. In what follows, "Γ" shall always be used to denote a functor whose domain is a small category, I. We shall also always use $A_i = \Gamma(i)$ for $i \in I$.

If $\Gamma : I \to \mathfrak{A}$ has an inf we shall denote it by $(A, u) = \inf \Gamma$ where $u = \{u_i : A \to A_i \mid i \in I\}$ is the required natural transformation from the constant functor to Γ.

If $\Gamma: I \to \text{Ens}$ then $\inf \Gamma = (A, u)$ always exists and we may assume that $A \subseteq \prod A_i$ and that each u_i is the restriction of the projection function $p_i : \prod A_i \to A_i$. It then follows that $x \in A$ iff $x \in \prod A_i$ and $h(p_i(x)) = p_j(x)$ whenever $h \in \Gamma(\text{Hom}(i, j))$.

LEMMA 1. *Let* $G : \mathfrak{A} \to \text{Ens}$ *be an inf-preserving functor whose action on morphisms is denoted by* $G(f) = \bar{f}$. *Let* F *be a function from the class of objects of* $\mathfrak{A}$ *to the class of sets. Assume* $F(A) \subseteq G(A)$ *for all* $A \in \mathfrak{A}$. *Then* F *can be regarded, in the natural way, as an inf-preserving functor iff*

(1) *for each morphism* $f : B \to A$ *it is true that*

$$\bar{f}(F(B)) \subseteq F(A);$$

(2) *whenever* $(A, u) = \inf \Gamma$, *for* $\Gamma : I \to \mathfrak{A}$, *then*

$$F(A) \supseteq \bigcap \bar{u}_i^{-1}(F(A_i)).$$

Proof. Clearly (1) is equivalent to the statement that F is functorial in the natural way. Notice that (1) and (2) imply $F(A) = \bigcap \bar{u}_i^{-1}(F(A_i))$. It suffices to show that $\inf(F\Gamma) = \bigcap \bar{u}_i^{-1}(F(A_i))$.

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