THE LOGARITHMIC EIGENVALUES OF PLANE SETS

BY

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Let S be a bounded set in the complex plane (E) having the same positive two dimensional Lebesgue measure as its closure $(\bar{S})$. Denote the open set complementary to $\bar{S}$ by $\tilde{S}$, and define S^*, the support of S, as follows:

$$S^* = \{z \in S : S \cap \Delta_r(z) \quad \text{has positive measure for each} \quad r > 0\}$$

where $\Delta_r(z)$ is the open disk of radius r and center z.

For such sets a functional μ is defined by

$$\mu(S) = \inf_{f \in L_1^2(S)} \left\{ -\frac{2}{\pi} \int_S \int_S \log |z - \zeta| f(z)\bar{f}(\zeta)\, d\tau_z\, d\tau_\zeta \right\}$$

where $L_1^2(S)$ is the set of all complex-valued functions which are square integrable over S with $\| f \|_2 \leq 1$, and τ is Lebesgue measure in the plane.

Clearly $\mu(S) \leq 0$, and in an earlier paper [1], it was shown that $\mu(S)$ is negative *iff* d, the transfinite diameter of $\overline{S^*}$ exceeds unity in which case the following inequality holds:

$$0 < -\mu < (2A/\pi) \log d$$

where A is the area of S^*.

Since $S \sim S^*$ has measure zero, it follows that $\mu(S) = \mu(S^*)$; hence attention may be restricted to bounded measurable support sets S, i.e. those plane sets for which $S = S^*$. (Observe that $(S^*)^* = S^*$.) Such sets which in addition have closures with transfinite diameter exceeding unity will be called admissible sets.

In the present paper, the dependence of μ on the class of admissible sets will be investigated. It will first be shown that μ is a monotone set functional which is continuous with respect to an appropriate type of convergence.

Next, a variational estimate for μ with respect to an important class of boundary variations is given and this formula is used to attack extremal problems suggested by the inequality: $0 < -\mu < (2A/\pi) \log d$. Specifically, it is proven that among all simply connected admissible domains of given transfinite diameter and sufficiently smooth boundary, the disk is the only one for which the value of the ratio $-\mu/A$ is stationary with respect to the boundary variations considered. Then, by use of specific domains, it is shown that $-\mu/A$ has neither maximum nor minimum under these conditions.

In [1], it was also shown that for admissible sets, μ is the unique negative

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