PERMUTATION REPRESENTATIONS

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A *permutation representation* of a group can be defined as a homomorphism of the given group G into a symmetric group, the group of all permutations of a given set Ω. We shall call the elements of Ω points and refer to Ω as a space, but this does not imply that any geometric notions are intended. We can also discuss permutation representations without talking about homomorphisms, by actually writing down the permutation for each g in G:

$$\varphi(g) = \binom{\omega}{\omega^g},$$

ω the general point of Ω. Everything is given once one is given the function ω^g of two variables; it will be called an *action* of G on Ω. This function is subject to the following requirements: for all ω in Ω, and g, h in G, we must have $(\omega^g)^h = \omega^{gh}$, and $\omega^1 = \omega$. These two ways of treating permutation representations are equivalent; I prefer to speak of actions because later we will also be talking about *linear representations.*

There are three major tools that have been developed for the purpose of studying the actions of a group G on a set Ω. The first of these is the well known theory of linear representations over a field, a theory developed by Frobenius around 1900.

The second method is due to Schur, and dates from 1933: this is the method of *Schur rings* [1], [3]. For the purposes of this paper we may define Schur rings in the following slightly simplified manner. Given a group H and a field F, consider the group ring FH. This is the ring of formal linear combinations of elements of H with coefficients in the field F, that is, the set

$$\{\sum a_h h : a_h \in F, h \in H\},$$

with coefficient-wise addition, and multiplication induced from the multiplication in H. A Schur ring is then a subring of FH, which is closed with respect to the additional operation of coefficient-wise multiplication, the operation defined by

$$(\sum a_h h)*(\sum b_h h) = \sum a_h b_h h.$$

Later we shall see that this peculiar type of operation turns up naturally in the theory of permutation representations.

There is one more method, of rather recent origin [4]. This is the study of those relations between points of Ω that remain invariant under the action of G. By studying these *invariant relations*, we hope to get information on the action of G on Ω. The particular case of binary relations can conveniently be represented by graphs. Graph theory has recently contributed considerably to the theory of permutation groups, e.g. in the work of Sims [2].