ALGEBRAIC GROUPS AND FINITE GROUPS

BY

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Our object is to indicate how large classes of finite simple groups, specifically those introduced by Chevalley [5], Suzuki [21], Ree [14], and us [16], can be studied profitably with the aid of the theory of linear algebraic groups. We shall refer to these groups as groups of Chevalley type, the first-mentioned as untwisted, the rest as twisted.

First we recall some facts about linear algebraic groups, all taken from [7]. Assume given an algebraically closed field k. A (linear) algebraic group is a subgroup of some $GL_n(k)$ which is at the same time an algebraic set (i.e. the complete set of solutions of a set of polynomials) in the space determined by the n^2 matric coefficients. The Zariski topology, in which the closed sets are the algebraic sets, is used. An algebraic group is said to be simple if it is connected, has only discrete, i.e. finite, nontrivial normal (algebraic) subgroups, and is not Abelian. The simple algebraic groups have been classified by Chevalley [7] in the Killing-Cartan tradition. Thus there are the classical types A_n, B_n, C_n, D_n and the five exceptional types E_6, E_7, E_8, F_4 and G_2. As examples we may mention the groups PSL_n or SL_n (of type A), SO_n or Spin_n (of type B or D depending on the parity of n), Sp_n (of type C), and the group of automorphisms of the Cayley algebra (of type G_2).

The connection between simple algebraic groups and simple finite groups comes from the fact that many of the latter, all of those mentioned in the first sentence above, arise as fixed-point groups of endomorphisms of the former.

The basic tool for studying the latter with the aid of the theory of algebraic groups is the following extension of a result of Lang [13].

(A) Let G be a connected linear algebraic group and σ an (algebraic) endomorphism of G onto G such that G_σ, the group of fixed points, is finite. Then the map $\varphi : G \rightarrow G$ defined by $\varphi x = x\sigma x^{-1}$ is surjective.

This is proved in [20]. Here we will sketch a proof in a special case, in which $\sigma x = x^{(q)}$, the result of replacing each entry of x by its q^{th} power, it being assumed that this operation maps G onto itself. Here q is a power of p, the characteristic of k, and is assumed to be greater than 1. From the rules of differentiation, the differential at $y = 1$ of the map $y \rightarrow y\sigma y^{-1}$ is the same as that of $y \rightarrow y$, hence is surjective. It follows that the map covers an open set in G. The same holds for the map $z \rightarrow zg\sigma z^{-1}$ with g an arbitrary element of G. Since G is connected, it is irreducible as an algebraic set [7, Exp. 3]. Thus the preceding open sets intersect: $y\sigma y^{-1} = zg\sigma z^{-1}$ for some y, z. Then $g = \varphi x$ with $x = z^{-1}y$, which proves (A), in this special case.

An immediate consequence of (A) is that if i_g is any inner automorphism, then $i_g\,\sigma$ is conjugate to σ: if $g = x\sigma x^{-1}$, then $i_g\,\sigma = i_x\,\sigma i_x^{-1}$. For the study of