A TRANSPLANTATION THEOREM FOR JACOBI SERIES

BY

RICHARD ASKEY[1]

1. Introduction

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, order (α, β), defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}],$$

$$\alpha, \beta > -1.$$

These polynomials are orthogonal on $(-1, 1)$ with respect to $(1-x)^\alpha(1+x)^\beta$ and

$$\int_{-1}^{1}[P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\,dx$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

$$= [t_n^{(\alpha,\beta)}]^{-2}.$$

Then the functions

$$\varphi_n^{(\alpha,\beta)}(\theta) = t^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}\,2^{(\alpha+\beta+1)/2}$$

are orthonormal functions on $(0, \pi)$ with respect to Lebesgue measure. The functions $\varphi_n^{(1/2,1/2)}(\theta)$ are $(2/\pi)^{1/2}\sin(n+1)\theta$, $n = 0, 1, \cdots$ and $\varphi_n^{(-1/2,-1/2)}(\theta) = (2/\pi)^{1/2}\cos n\theta$, $n = 1, 2, \cdots$, $\varphi_0^{(-1/2,-1/2)}(\theta) = \pi^{-1/2}$. Fourier series with respect to these two sets of functions have been studied extensively. Fourier series for Jacobi polynomials have not been as extensively studied and many fewer results are known. The one type of result that has been studied in any detail deals with equiconvergence theorems. This type of result comes from asymptotic formulas for $\varphi_n^{(\alpha,\beta)}(\theta)$ which are valid for $\varepsilon \leqq \theta \leqq \pi - \varepsilon$. However, for many of the results in Fourier analysis we want to use all the values of θ, $0 \leqq \theta \leqq \pi$. In this paper we show how to set up a bounded mapping between Fourier series with respect to Jacobi polynomials which allows one to read off many of the deep results for Fourier-Jacobi expansions from the corresponding results for ordinary Fourier series.

I haven't discussed this work with Stephen Wainger, but many of the ideas that are used arose in connection with other work we have done together and I would like to acknowledge my indebtedness to these discussions.

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