CARTER SUBGROUPS AND FITTING HEIGHTS OF FINITE SOLVABLE GROUPS

BY

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Let G be a finite solvable group having Fitting height h (as defined in [7] or in §1 below). Let H be a Carter subgroup of G and l be the length of a composition series of H. We shall establish the correctness of a conjecture of John Thompson (at the end of [7]) by proving that

$$h \leq 10(2^l - 1) - 4l. \tag{0.1}$$

This is the result of Theorem 8.5 below, and the rest of this paper is a proof of that theorem.

The upper bound for h given by (0.1) is almost certainly too large. The work of Shamash and Shult [6] leads one to conjecture that there is some constant K such that

$$h \leq Kl, \tag{0.2}$$

for all finite solvable groups G. The methods of this paper unfortunately cannot give an upper bound whose order of magnitude is less than 2^l. This is caused by our very naive approach. Essentially we choose a normal subgroup P of prime order in H and a suitable chain $A_1, \cdots, A_h$ of H-invariant sections of G. Obviously either P centralizes $A_1, \cdots, A_{[h/2]}$ or there exists a subchain $A_k, A_{k+1}, \cdots, A_{k+[h/2]}$ such that P does not centralize A_k. In the latter case we construct (and this is the hard part of the proof) an H-invariant chain $D_{k+j}, D_{k+j+1}, \cdots, D_{k+[h/2]}$ of sections of $A_{k+j}, A_{k+j+1}, \cdots, A_{k+[h/2]}$ (respectively) such that j is bounded and P centralizes each D_i. In either case we obtain a chain of length "almost" $h/2$ of sections of G on which H/P acts, and which satisfies suitable axioms so that the process can be repeated (using a normal subgroup of prime order in H/P, etc.) Obviously no method based on this process can give an upper bound smaller than 2^l.

There are many technical complications in the proof due to the difficulty of handling the case $|P| = 3$ (among other things). But basically it is a straightforward application of the methods of Hall and Higman [3]. The few new concepts which are used are grouped together in Sections 1, 2 and 3. They are the notions of *Fitting chains* (which are the "correct" chains of sections $A_1, \cdots, A_n$ of G), of *weak equivalence* (which is used in place of equivalence in Fitting chains because it is impossible to verify the latter after

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