# ASYMPTOTIC DISTRIBUTION OF BEURLING'S GENERALIZED INTEGERS

BY

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## 0. Introduction

This paper is a study of the asymptotic distribution of Beurling's generalized numbers which improves previous estimates of the error term. Our approach to the problem is first to show how measures of a certain type may be expressed as "exponentials." We then apply this formalism to the counting measure of generalized integers and Lebesgue measure and estimate the one measure by the other. With this representation we show how a hypothesis on the distribution of generalized primes leads to an estimate of the distribution of generalized numbers.

Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of real numbers subject to the following three conditions but otherwise arbitrary:

$$\text{(i) } p_1 > 1, \qquad \text{(ii) } p_{n+1} \geq p_n, \qquad \text{(iii) } p_n \to \infty.$$

Following Beurling [1] we call such a collection $\{p_i\}$ a set of generalized (henceforth $g$-) primes. The multiplicative semigroup generated by the $\{p_i\}$ is countable and may be arranged in a nondecreasing sequence $\{n_i\}_{i=1}^{\infty}$. Setting $n_0 = 1$, we call $\{n_i\}_{i=0}^{\infty}$ the set of $g$-integers associated with $\{p_i\}$.

The function $\pi(x)$ is defined to be the number of $g$-primes less than or equal to $x$, and $\Pi(x)$ is defined by

$$\Pi(x) = \textstyle\sum_{n=1}^{\infty} (1/n)\pi(x^{1/n}).$$

Let $N(x)$ be the number of $g$-integers less than or equal to $x$.

With these definitions, our main results take the following form:

THEOREM. *Suppose there exist positive numbers $c$ and $\alpha$ such that for large $x$ the following relation holds:*

$$\int_1^x d\Pi(t)/t = \int_1^x (1 - t^{-1})\, dt/(t \log t) + \log c + O(\log^{-\alpha} x).$$

*Then* $N(x) = cx + O(x \log^{2-\alpha} x)$.

THEOREM. *Suppose there exist numbers $c > 0$ and $a \in (0, 1)$ such that*

$$\int_1^x d\Pi(t)/t = \int_1^x (1 - t^{-1})\, dt/(t \log t) + \log c + O\{\exp(-\log^a x)\}.$$

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