EXTENDED BASES FOR BANACH SPACES

BY

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1. Introduction

Let E be a Banach space over the real or complex field. A sequence $\{x_i\}$ in E is said to be a (weak) basis for E if for every x in E there corresponds a unique scalar sequence $\{\alpha_i\}$ such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$, the convergence being in the norm (weak) topology of E. A basis with continuous coefficient functionals α_i is called a Schauder basis. In [2], Arsove and Edwards introduced the concept of an *extended* Schauder basis, where, by discarding the requirement of countability, they carried out the expansion according to a given directed set. Without proof they have given the following theorem: Every weak extended Schauder basis for E is an extended Schauder basis for E. As already indicated in [2], it is usually assumed that the expansions converge unconditionally, and so we obtain a slightly stronger definition of an extended basis: A family $\{x_\lambda\}$ $(\lambda \in \Lambda)$ is said to be a (weak) extended unconditional basis, or, in short, a *(weak) extended basis* for E if to each x in E there is a unique scalar family $\{\alpha_\lambda\}$ such that $x = \lim_\sigma \sum_{\lambda \in \sigma} \alpha_\lambda x_\lambda$ in the norm (weak) topology of E, where the σ's are finite subsets of Λ, directed by inclusion, and where $\lim_\sigma y_\sigma$ denotes the limit of a net $\{y_\sigma\}$ in E. If, according to Bessaga and Pelczynski [3], an absolute basis for E denotes a total set in E in which every sequence of distinct elements forms an unconditional basic sequence, it turns out that the extended bases for E and the absolute bases for E are the same. Moreover, it is easy to see that in a separable space the concepts of extended and unconditional bases are the same. But the example of a (non-separable) Hilbert space shows that there exist extended bases which are not bases. However, it is known [3] that there exist Banach spaces, e.g. l_∞, which have no absolute and hence no extended bases; a negative fact which is not yet cleared for bases. Since l_∞ is an $\mathfrak{N}_p$-space $(p > 1)$ there are $\mathfrak{N}_p$-spaces without extended bases (for the definition see [8]).

It is encouraging that many results from the theory of bases in separable spaces have their analogues for extended bases. Indeed, as shown in this note, one can establish necessary and sufficient conditions for E to have an extended basis, a theorem which formally resembles the theorem of Nikol'skiĭ which applies to bases for separable Banach spaces. It is also shown that the weak extended bases coincide with the extended bases and that the extended bases coincide with the extended Schauder bases. Using the natural extensions of the notions of shrinking or boundedly complete bases one obtains a

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