# EUCLIDEAN AND NON-EUCLIDEAN NORMS IN A PLANE

BY

D. A. SENECHALLE

*Introduction.* Let $L$ denote a 2-dimensional linear space. If $f$ is any norm on $L$, $K(f)$ denotes the smallest number $r > 0$ such that for some Euclidean norm $g$ dominated by $f$, the norm $r \cdot g$ dominates $f$. Note that $K(f) = 1$ or $K(f) > 1$ according as $f$ is Euclidean or not. The following are the main results in this paper: (1) that

$$K(f) = \sup \{[(f^2(ax + y/a) + f^2(bx - y/b))/(a^2 + 1/a^2 + b^2 + 1/b^2)]^{1/2}: \\ a, b > 0, f(x) = f(y) = 1\};$$

(2) a theorem which shows how to construct all norms $f$ with $K(f)$ fixed; (3) some improvements on known conditions for inner product spaces with the change that they are required to hold only locally or in the limit.

*Notation and preliminaries.* For any linearly independent $x$ and $y$ in $L$, $C(x, y)$ denotes the set $\{ax + by: a, b \geqq 0\}$ and $W(x, y)$ denotes the set $\{ax + by: ab \geqq 0\}$. We call a quadruple of points $(x, y, x', y')$ interlocking if the points are pairwise linearly independent, $C(y', y) \supset C(x, y) \supset C(x', y)$, and the unit sphere of some norm contains them. If $f$ is any functional on $L$ define $S(f)$ and $U(f)$ to be $f^{-1}(1)$ and $f^{-1}([0, 1])$ respectively. Define a subnorm to be any restriction $f$ of a norm on $L$ such that dom $f$ is closed, $R \cdot \text{dom } f = \text{dom } f$, and there exists an interlocking quadruple of points of $S(f)$. Call a functional $f$ on $L$ a Euclidean pre-norm if either $f$ is a Euclidean norm or $f = |g|$ for some $g \neq 0$ in $L^{\#}$. If $f$ is any subnorm, $E_f$ ($E^f$) denotes the set of all Euclidean pre-norms dominating (dominated by) $f$ over dom $f$, and if $g$ is in $E_f$ of $E^f$, $d(f, g)$ denotes

$$\sup \{g(x), 1/g(x) : x \in S(f)\}.$$

If $N$ is $E_f$ or $E^f$, $d(f, N)$ denotes $\inf_{g \in N} d(f, g)$. Note that the definition of $K(f)$ can be extended to subnorms by an obvious modification.

If $w = (x, y, x', y')$ is any interlocking quadruple, define

$$k(w) = [(ab + cd)/(cd(a^2 + b^2) + ab(c^2 + d^2))]^{1/2},$$

where $x' = ax + by$ and $y' = cx - dy$. Thus $(a, b, c, d > 0)$. We list without proof the following four properties of any interlocking quadruple $w = (x, y, x', y')$:

($P_1$) There exists only one ordered pair $(r, C)$ such that $r > 0$, $C = S(f)$ for some Euclidean pre-norm $f$, $C$ contains $x'$ and $y'$, and $r \cdot C$ contains $x$ and $y$.

($P_2$) $r = k(w)$.

Received December 23, 1968.