THE RING OF POLAR PRESERVING ENDOMORPHISMS OF AN ABELIAN LATTICE-ORDERED GROUP

BY

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1. Introduction

Let G be an abelian lattice-ordered group (l-group). We investigate the ring $P(G)$ generated by the semiring $P^+(G)$ of all group endomorphisms α of G such that for $x, y \in G$

$$x \wedge y = 0 \quad \text{implies} \quad x \wedge y\alpha = 0.$$

$P(G)$ is a po subring of the ring $B(G)$ of all order-bounded endomorphisms of G with $P^+(G)$ as its positive cone. If A is any ring of l-endomorphisms of G that contains the identity automorphism I, then $A \subseteq P(G)$. Thus $P(G)$ is the largest such ring. We show (Theorem 3.4) that the class

$$\{P(G) : G \text{ is an archimedean } l\text{-group}\}$$

is identical with the class of archimedean f-rings with identity. This allows us to derive many useful properties of $P^+(G)$.

For an archimedean l-group G, the largest f-ring of $B(G)$ that contains the identity is $P(G)$. Let G be an archimedean l-group with a weak order unit e. Then there is at most one multiplication on G so that G is an f-ring with identity e, and such a multiplication exists if and only if $\{e\alpha : \alpha \in P^+(G)\} = G^+$.

The elements in $P^+(G)$ preserve minimal prime subgroups. In Section 6 we investigate those group endomorphisms of G which preserve all the prime subgroups. In Section 7 we apply our theory to solve a problem posed by G. Birkhoff.

Notation and terminology. If G is an l-group, then we denote its positive cone by $G^+ = \{g \in G : g \geqq 0\}$. An *$l$-subgroup* of G is a subgroup K which is also a sublattice. If, in addition, $0 < x < k \in K$ implies $x \in K$, then we say that K is a *convex l-subgroup*. An *l-ideal* is a normal convex l-subgroup. A *prime* subgroup is a convex l-subgroup M such that $x \wedge y \in M$ implies $x \in M$ or $y \in M$. Various other characterizations of prime subgroups are given in [4] and [9]. An *l-endomorphism* of G is a group endomorphism that also preserves the lattice operations. Thus an endomorphism α of G is an l-endomorphism if and only if $x \wedge y = 0$ implies $x\alpha \wedge y\alpha = 0$ [7].

If X is a subset of G, then

$$X' = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called the *polar* of X. X' is a convex l-subgroup of G and the set $p(G)$ of

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