# ON THE INVERSE FUNCTION THEOREM IN COMMUTATIVE BANACH ALGEBRAS

BY

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## Introduction

Let $A$ be a complex commutative Banach algebra, and $D$ a domain in $A$. An analytic isomorphism of $D$ is an injective, $L$-analytic (i.e. analytic in the sense of Lorch [4]) mapping $f : D \to A$ so that $f(D)$ is also a domain, and $f^{-1}$ is $L$-analytic on $f(D)$. It is known that if $f : D \to A$ is $L$-analytic and $f'(a_0)$ is invertible, then there is some open neighborhood $U$ of $a_0$ in $D$ so that $f \mid U$ is an analytic isomorphism of $U$. This result sets the classical inverse function theorem for analytic functions of a complex variable in the Lorch theory of analytic functions of an $A$-variable. It is an immediate consequence of the remarks of Arens and Calderon [2, p. 214] on the inversion of a power series with coefficients in $A$, and was first explicitly given by Mibu [5, p. 333].

The central goal of this paper is to prove the following two theorems, which are both related to, and corollaries of, the above inverse function theorem.

THEOREM 1. *If $f : D \to A$ is $L$-analytic and injective, $f(D)$ is a domain, and $f^{-1}$ is continuous on $f(D)$, then $f$ is an analytic isomorphism of $D$.*

THEOREM 2. *Suppose $A = C(X)$, where $X$ is a compact Hausdorff space. If $f : D \to A$ is $L$-analytic and injective, then either $f$ is an analytic isomorphism of $D$, or there is some fixed $x \in X$ so that $f(g)(x)$ is identically constant, all $g \in D$.*

In a preliminary section, we discuss the quotient function $f_F$ (which may or may not exist) and the general quotient (possibly multiple-valued) function $f^F$ (which always exists) of an $L$-analytic $f : D \to A$ by a maximal ideal $F$ of $A$. Both $f_F$ and $f^F$ will be used in the proofs of Theorems 1 and 2, and are of interest in their own right. In this regard, we will prove that if $D$ is star-shaped, then $f_F$ exists, and then give an example where $f_F$ does not exist even though $D$ is simply connected.

The author would like to raise the following questions.

(a) Can the hypothesis that $f^{-1}$ be continuous be removed from Theorem 1?

(b) Can Theorem 2 be generalized to other Banach algebras?

## Notation and terminology

1. $A$ will denote a complex, commutative Banach algebra with identity.
2. $D$ will denote a domain in $A$, i.e. an open, connected subset of $A$.
3. $D$ is simply connected iff each loop in $D$ is homotopic to a point in $D$.

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