INVARIANTS FOR COMMUTATIVE GROUP ALGEBRAS

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Let K be a commutative ring with identity and G an abelian group. Then the structure of KG as a K-algebra depends to some extent upon the primes p for which the torsion subgroup of G has non-trivial p-components and the relationship of these primes to the arithmetic of K. The case in which these primes are not invertible in K has been investigated in [2] and it was seen that the algebraic structure of these p-components is intimately connected with that of the algebra. If the ring K is especially nice, namely an integral extension ring of the integers, then it is shown in [3] that the isomorphism class of KG determines the isomorphism class of G, hence this latter class is a complete set of invariants for commutative group algebras over K.

In this paper we consider a case at the opposite extreme. Take K to be an algebraically closed field and G an abelian group having no element whose order is equal to the characteristic of K. Then all primes of the type mentioned above are invertible in this ring and so we should expect the structure of KG to be related only weakly to that of G. Of course when G is finite it is well known that KG is isomorphic to the direct product of n copies of K where n is the cardinality of G, hence in the finite case the cardinality of G (or the dimension of KG) constitutes a complete set of invariants. We shall show that in general, a complete set of invariants for the structure of KG consists of the cardinality of G_0 and the isomorphism class of G/G_0 (where G_0 is the torsion subgroup of G). Moreover we shall say something about how these invariants can be determined from the algebra.

For the rest of this paper, K will denote an algebraically closed field. In addition we shall tacitly assume that every group considered will have no element of order equal to the characteristic of K.

PROPOSITION. *Let G be an abelian group with torsion subgroup G_0. Then*

$$KG \cong KG_0 \otimes_K K(G/G_0).$$

Proof. Define $H = G_0 \times (G/G_0)$. Since $KH \cong KG_0 \otimes_K K(G/G_0)$, it will suffice to show that $KG \cong KH$. This will be accomplished by finding a group of units in KH which is isomorphic to G and is a K-basis for KH. First we must choose a certain generating set for G.

We wish to construct a family of subgroups of G, $\{G_\alpha\}$, indexed by some initial segment of ordinals. Start with G_0. Define $G_{\alpha+1}$ to be the subgroup generated by G_α and an element $g_\alpha \notin G_\alpha$ in case $G_\alpha \neq G$. If α is a limit ordinal, define $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Then $G = \bigcup_\alpha G_\alpha$. Now for each α, let n_α be 0 in case $\langle g_\alpha \rangle \cap G_\alpha = \{1\}$, otherwise let n_α be the least positive integer such that

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