VECTOR LATTICES AND SEQUENCE SPACES[1]

BY

G. Crofts and J. J. Masterson

1. Introduction

In [6], Peressini and Sherbert study Köthe sequence spaces and mappings between these spaces with special emphasis on their order properties. Variations of these results appear in [5]. The purpose of this paper is to extend some of these results. Among other things, it will be shown by using techniques involving the Dedekind completion of a vector lattice, that condition "E is Dedekind complete" can be replaced by "E is Archimedean" in several places.

Let E be a vector lattice. We say the net $\{x_\alpha\}$ order converges to x in E (we write $x_\alpha \to^\circ x$) if there exists a net $\{y_\alpha\} \subset E$ such that $|x_\alpha - x| \leqq y_\alpha$ and $y_\alpha \downarrow 0$ (i.e., $\{y_\alpha\}$ is down-directed and $\inf(y_\alpha) = 0$). As in [6], we note the following significant restrictions which may be put on E.

(A) If A is a subset of E that has a supremum, then there is a countable subset A' of A such that $\sup A' = \sup$ A. (We say that E is *order separable.*)

(B) If $\{y_{n,m}\}$ is a sequence in E that order converges to y_n in E (for each $n = 1, 2, \cdots,$) and if $\{y_n\}$ order converges to y_0 there is an increasing sequence $\{m_n : n = 1, 2, \cdots,\}$ of positive integers such that $\{y_{n, m_n}\}$ order converges to y_0. (We say E has the *diagonal property.* Note that we use the definition in [5] rather than [6].)

(C) If $\{A_n\}$ is a sequence of non-majorized subsets of E, there exist finite subsets A'_n of A_n $(n = 1, 2, \cdots)$ such that

$$\{\sup A'_n : n = 1, 2, \cdots\}$$

is not a majorized set. (We say that E is *finitely unbounded.*)

2. The Dedekind completion

Let E be an Archimedean vector lattice throughout this section. Nakano has shown [3] that a necessary and sufficient condition that E be Archimedean is that it possess a Dedekind completion (or cut completion) i.e., that there exist a complete vector lattice $\hat{E}$ such that:

(D1) E is embedded as a subvector lattice in $\hat{E}$,

(D2) for each $u \in \hat{E}$,

$$u = \sup\{x : x \in E, x \leqq u\} = \inf\{y : y \in E, y \geqq u\}$$

In [1], it is shown that (D2) can be replaced by

(D2′) For $0 < u \in \hat{E}$, there are x, y in E such that $0 < x \leqq u < y$.

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